Problem Set 1 (due Wednesday, September 24)

1. **Applying low-pass and bandpass filters to a digital signal**

A square periodic signal is represented as the following sum of sinusoids:

\[
s(t) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos((2k+1)\pi t).
\]

(Note that this is just a rewriting of the formula we discussed in class.)

(a) Suppose that the signal is applied to an ideal low-pass filter with bandwidth 15 Hz. Plot the output from the low-pass filter and compare to the original signal. Repeat for 5 Hz; for 3 Hz. What happens as the bandwidth increases. [5]

(b) Suppose that the signal is applied to a bandpass filter that passes the frequencies from 5 to 9 Hz. Plot the output from the filter and compare to the original signal. [5]

For your plots, use an appropriate plotting tool. One such tool is gnuplot, available in Unix.

2. **Fourier Transforms**

(a) Show that convolution in the time domain is multiplication in the frequency domain. In other words show that if

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) n(t - \tau) d\tau
\]

then

\[
Y(f) = X(f) N(f)
\]

[3]

(b) Consider the signal \( x(t) = e^{-at}, a > 0 \). Plot the phase and magnitude of the Fourier transform as a function of \( f \). [4]

(c) Explain why the Fourier basis of complex exponentials is a good basis for representing (wireless) signals. In particular, why are complex exponentials superior to sinusoidal? [3]

3. **Sampling**

(a) State and prove the Nyquist sampling theorem [5]
(b) Consider the signal \( x(t) = \cos(\frac{f_s}{2}t) \). If the sampling frequency is \( f_s \) then what does the sampled signal look like? What does the reconstructed signal look like? Plot both. Why are we unable to reconstruct the original signal? [5]

4. Shannon’s theorem

(a) A digital signaling system is required to operate at 38.4 Kbps. If a signal element encodes a 8-bit word, what is the minimum required bandwidth of the channel. What signal-to-noise ratio is required to achieve the desired capacity on the bandwidth that you have computed? [4]

(b) Derive the “spectral efficiency” form of the Shannon theorem from the capacity version. Plot the spectral efficiency (bits per Hertz) in terms of the bit-energy-to-noise-density. Indicate the attainable region. Calculate the largest value (in dB) of bit-energy-to-noise-density below which there can be no error-free communication. [6]

5. Antenna

(a) Show that doubling the transmission frequency is equivalent to doubling the distance between the sending and receiving antennae. Calculate, in dB, attenuation of power in either case. [4]

(b) Suppose we used the thumbrule that the optical line of sight distance from an antenna \( h \) meters high to the horizon, in kilometers, is \( 4\sqrt{h} \). What is the radius of the earth in kilometers that would justify this approximation? Assume the earth to be a perfect sphere. [3]

(c) Assume that two antennas are half-wave dipoles, each with a directive gain of 10 dB. If the transmitted power is 0.5W and the two antennae are separated by a distance of 5km, what is the received power? Assume perfect alignment of the antennae and a frequency of 300MHz. [3]