College of Computer Science | Lecture 2
Northeastern University | September 15 2008
Wireless Networks CS G250 | Lecturer: R. Sundaram

Fourier Transforms, Sampling

Handouts: Term Project Information
You can find information about the final project at
`http://www.ccs.neu.edu/course/csg250/TermPaperInformation-4.htm`
Lecture Outline:

- Fourier Transforms
- Sampling

1 Fourier Transforms

1.1 Linear Algebra

A vector can be thought of as a point in n-dimensional space. In 2-dimensional space, for instance, the vector $\binom{4}{3}$ is a point in plane that is 4 units on the x-axis and 3 units on the y-axis. The length (also called magnitude or norm) of this vector is 5. Assume now that a certain system stretches the vector such that its distance from the origin doubles (Figure 1). The streched vector is $\binom{8}{6}$ and the length of the streched vector is 10. Notice, however, that the streching operation "spills over" both of the vector coordinates, making this operation cumbersome.

In order to simplify the streching operation, we could use a different coordinate system. The new coordinate system will be related to the original coordinate system and will have the property that the original vector lies along the x'-axis of the new coordinate system (Figure 2).

In the new coordinate system, during the streching operation the vector will have only its x value changed, therefore making the streching operation a lot simpler.

1.2 Noise and the Frequency Domain

Suppose a certain signal $x(t)$ is to be transmitted. The signal sent is not equal to the signal seen on the receiving end, because of noise. We will call

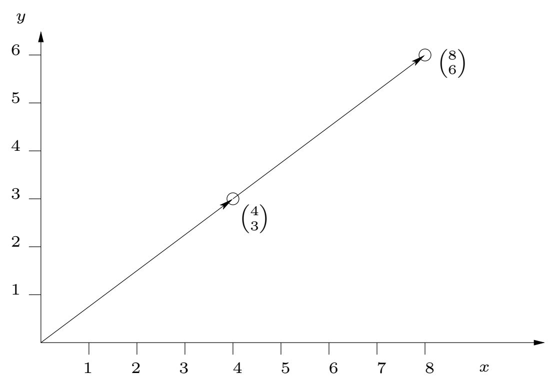

Figure 1: A vector and the resulting vector after stretching

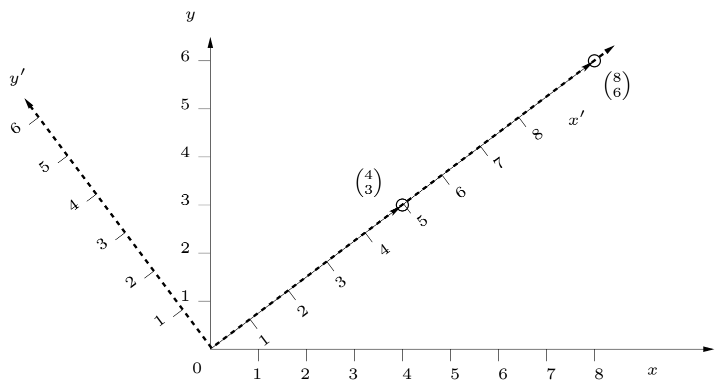

Figure 2: Vector and shifting. The new coordinate system is shown in thicker, dashed lines

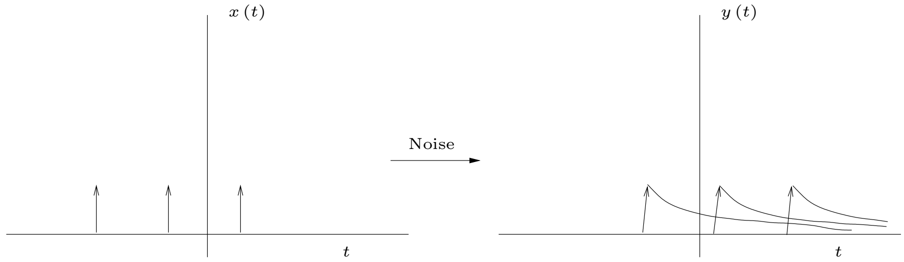

Figure 3: Dirac delta pulses at frequency f, and the received signal

the signal received $y(t)$. By applying the concept of vectors, any signal can be considered an ∞-dimensional vector. We would like the noise to become natural and localized to one coordinate.

When a pulse is transmitted, at the receiving end, the pulse is corrupted with noise. The receiver will see a slight slant for the pulse and the signal will decay at a certain rate afterwards.

Consider Figure 3 that shows the transmission of a noise signal $n\,(t-\tau)$ that consists of impulses of unit size at a certain frequency. In a $\infty-$dimensional basis, this translates into an operation that "spills" every coordinate after τ, analogous to what happens in Figure 1. The function $n\,(t)$ is an example of a Linear Time-Invariant system (LTI for short; linear as in additive and time invariant as in independent of time).

Let's consider the function $\delta\,(t)$ defined as the impulse of unit size at $t = 0$ and 0 everywhere else. So, a signal can be defined as:
$x\,(t) = \int_{-\infty}^{\infty} x\,(\tau)\,\delta\,(t-\tau)\,d\tau$.

The effect of noise on signal $x(t)$ is

$$y(t) = \int_{-\infty}^{\infty} n(\tau)\,x_f(t-\tau)\,d\tau$$

and we would like to find a function x_f such that $y(t) = cx_f(t)$ for some constant c. In other words, we need an Eigen function of x and we will call it x_f.

Complex exponentials constitute an eigenbasis for this case. Let

$$\begin{aligned}
x_f(t) &= e^{i2\pi ft} \\
x(t) &= \int_{-\infty}^{\infty} n(\tau)x_f(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} n(\tau)e^{i2\pi(t-\tau)f}d\tau \\
&= e^{i2\pi ft}\int_{-\infty}^{\infty} n(\tau)e^{-i2\pi f\tau}d\tau
\end{aligned}$$

We will define $N(f) = \int_{-\infty}^{\infty} n(\tau)e^{-i2\pi f\tau}d\tau$. This is the Fourier Transform of the n function in the frequency domain.

To understand what happens to functions that undergo this transformation, let's consider the step function in the interval $[-T, T]$ shown in Figure 4.

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi tf}dt \\
&= \int_{-T}^{T} e^{-i2\pi tf}dt \\
&= \left.\frac{e^{i2\pi ft}}{2\pi f}\right|_{-T}^{T} \\
&= \frac{sin2\pi fT}{\pi f}
\end{aligned}$$

The transformed function is shown in Figure 5.

In conclusion, the properties of a basis should:

1. represent a large class of functions
2. be an LTI eigen function

Not all functions can be transformed in the way presented in this section, however. For a function to be representable in the frequency domain, the following conditions (dubbed *Dirichlet's conditions*) must be satisfied

1. The function must be absolutely integrable
2. The function must have a bounded variation

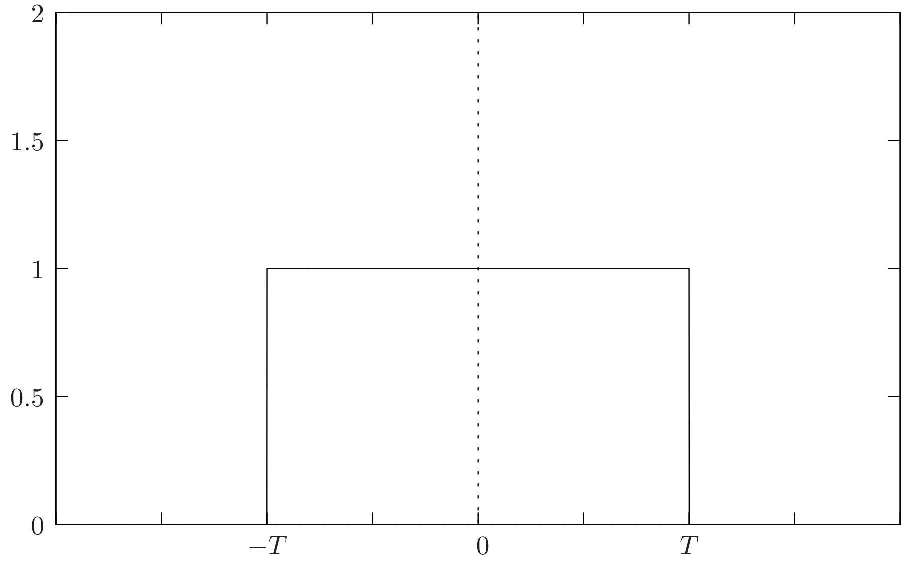

Figure 4: The box function between $-T$ and T

3. The function must have a finite number of discontinuities

A notable property of the above transformation is the equivalence of convolution to multiplication. That is to say:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} n(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} N(f)X(f)e^{i2\pi ft}df \\ Y(f) &= N(f)X(f) \end{aligned}$$

Additionally, *Parsevalis equality* relates the total energy associated with both functions:

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df = E$$

2 Sampling

Suppose you measure a signal at intervals of time T. Is there any way of knowing what the original signal looked like using only the measured points?

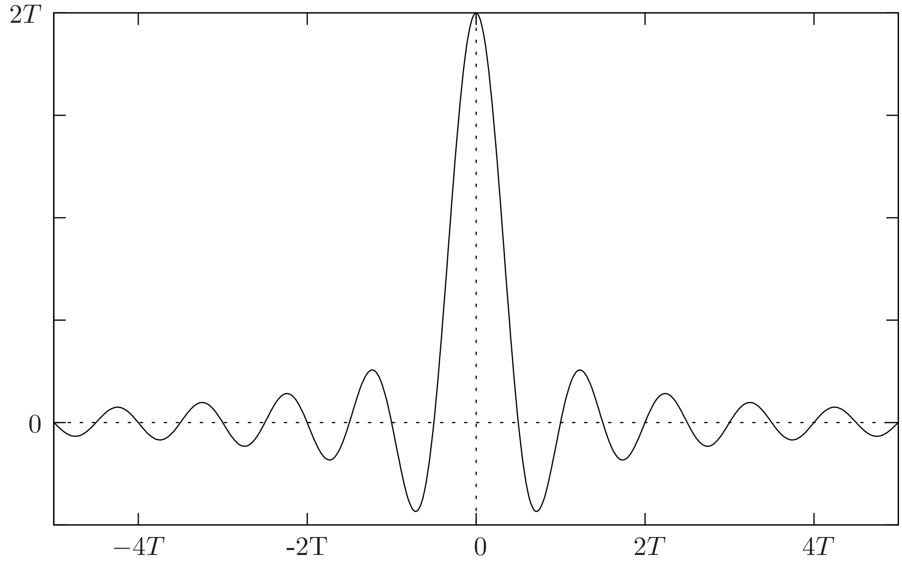

Figure 5: The box function in the frequency domain

Intuitively, there are infinitely many possibilities for functions that may pass through the measured, discrete points. However, if we restrict how much the original signal oscillates in time, the original signal may be reconstructed.

Theorem 1 *Nyquist Sampling Theorem*
Let $x(t)$ be a signal whose frequency components are no greater than a certain frequency f_B. Then the original signal can be completely described from samples taken at a rate $f_s > 2f_B$

To sample a signal $x(t)$, we multiply it by a sampling signal $p(t)$, whose value is 1 when the sample is taken and 0 otherwise. So $x_s(t) = x(t)p(t)$ where $x_s(t)$ is the resulting sampled signal.
The sampling signal is periodic, so we can write it as

$$p(t) = \sum_{n=0}^{\infty} P_n e^{i2\pi nft}$$

for some values of P_n.

What does the Fourier Transform of $x_s(t)$ look like? If we assume a sampling frequency of f_s

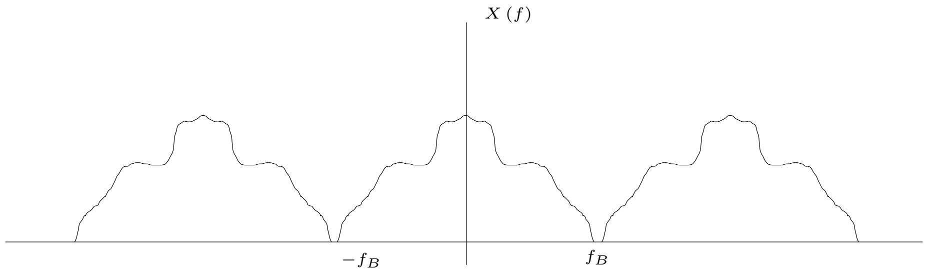

Figure 6: Sampled signal in the frequency domain. The original signal lies completely between $-f_B$ and f_B

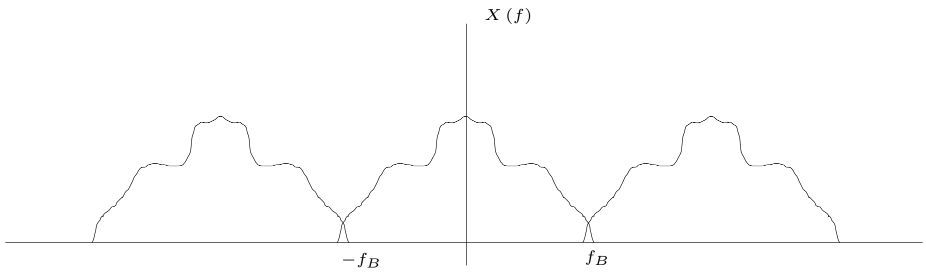

Figure 7: A signal sampled with a frequency $f < 2f_B$ will generate a graph in the frequency domain where the ends overlap

$$
\begin{aligned}
X_s(f) &= \int_{-\infty}^{\infty} x_s(t)e^{-i2\pi ft}dt \\
&= \int_{-\infty}^{\infty} x(t)\sum_{n=0}^{\infty} P_n e^{i2\pi nf_s t}e^{-i2\pi ft}dt \\
&= \sum_{n=0}^{\infty} P_n \int_{-\infty}^{\infty} x(t)e^{i2\pi(nf_s-f)t}dt \\
&= \sum_{n=0}^{\infty} P_n \int_{-\infty}^{\infty} x(t)e^{-i2\pi(f-nf_s)t}dt \\
&= \sum_{n=0}^{\infty} P_n X(f-nf_s) \\
&= P_0X(f)+P_1X(f-f_s)+P_2X(f-2f_s)+...
\end{aligned}
$$

The Fourier Transform of the sample signal is made of "copies" of the Fourier Transform of the original signal located at f_s intervals away. Since the original signal has no frequency components over f_B, then its figure in the frequency domain looks like the central shape in Figure 6, which lies entirely between $-f_B$ and f_B. Then, when $f_s > 2f_B$, the copies will not overlap.

When $f_s \leq 2f_B$, the ends will overlap, making it impossible to retrace the original values of the signal. This is called undersampling and a signal obtained from such a processed is said to be *aliased*. There are several examples of this:

1. A film that shows the wheels of a car as if they were turning backwards
2. A photograph in which the details of a pattern lie below the resolution of the imaging device