Lecture #11 Overview

- Vector representation of signal waveforms

- Two-dimensional signal waveforms

Geometric Representation of Signals

- We shall develop a *geometric representation* of signal waveforms as points in a signal space.

- Such representation provides a *compact characterization* of *signal sets* for transmitting information over a channel and simplifies the *analysis* of their *performance*.

- We use vector representation which allows us to represent *waveform communication channels by vector channels*.

Geometric Representation of Signals

- Suppose we have a set of M signal waveforms $s_m(t), 1 \leq m \leq M$ where we wish to use these waveforms to transmit over a communications channel (recall QAM, QPSK).

- We find a set of $N \leq M$ *orthonormal basis* waveforms for our *signal space* from which we can construct all of our M signal waveforms.

- *Orthonormal* in this case implies that the set of basis signals are orthogonal (inner product $\int s_i(t)s_j(t)dt = 0$) and each has *unit energy*.

Orthonormal Basis

- Recall that $\tilde{i}, \tilde{j}, \tilde{k}$ formed a set of orthonormal basis vectors for 3-dimensional vector space, $\mathbb{R}^3$, as any possible vector in 3-D can be formed from a *linear combination* of them:
  $\tilde{v} = v_i\tilde{i} + v_j\tilde{j} + v_k\tilde{k}$

- Having found a set of waveforms, we can express the M signals $\{s_m(t)\}$ as exact *linear combinations* of the $\{\psi_j(t)\}$

$$s_m(t) = \sum_{j=1}^{N} s_{mj}\psi_j(t) \qquad m = 1, 2, \ldots, M$$

where

$$s_{mj} = \int_{-\infty}^{\infty} s_m(t)\psi_j(t)dt$$

and

$$\mathcal{E}_m = \int_{-\infty}^{\infty} s_m^2(t)dt = \sum_{j=1}^{N} s_{mj}^2$$

Vector Representation

- We can therefore represent each signal waveform by its vector of coefficients s_{mj}, knowing what the basis functions are to which they correspond.

$$\boldsymbol{s_m} = [s_{m1}, s_{m2}, \ldots, s_{mN}]$$

- We can similarly think of this as a point in N-dimensional space

- In this context the *energy of the signal waveform* is equivalent to the *square of the length of the representative vector*

$$\mathcal{E}_m = |\boldsymbol{s_m}|^2 = s_{m1}^2 + s_{m2}^2 + \ldots + s_{mN}^2$$

- That is, the *energy* is the square of the *Euclidean distance* of the point $\boldsymbol{s_m}$ from the origin.

Vector Representation (cont.)

- The *inner product* of any two signals is equal to the *dot product* of their vector representations

$$\boldsymbol{s_m} \cdot \boldsymbol{s_n} = \int_{-\infty}^{\infty} s_m(t) s_n(t) dt$$

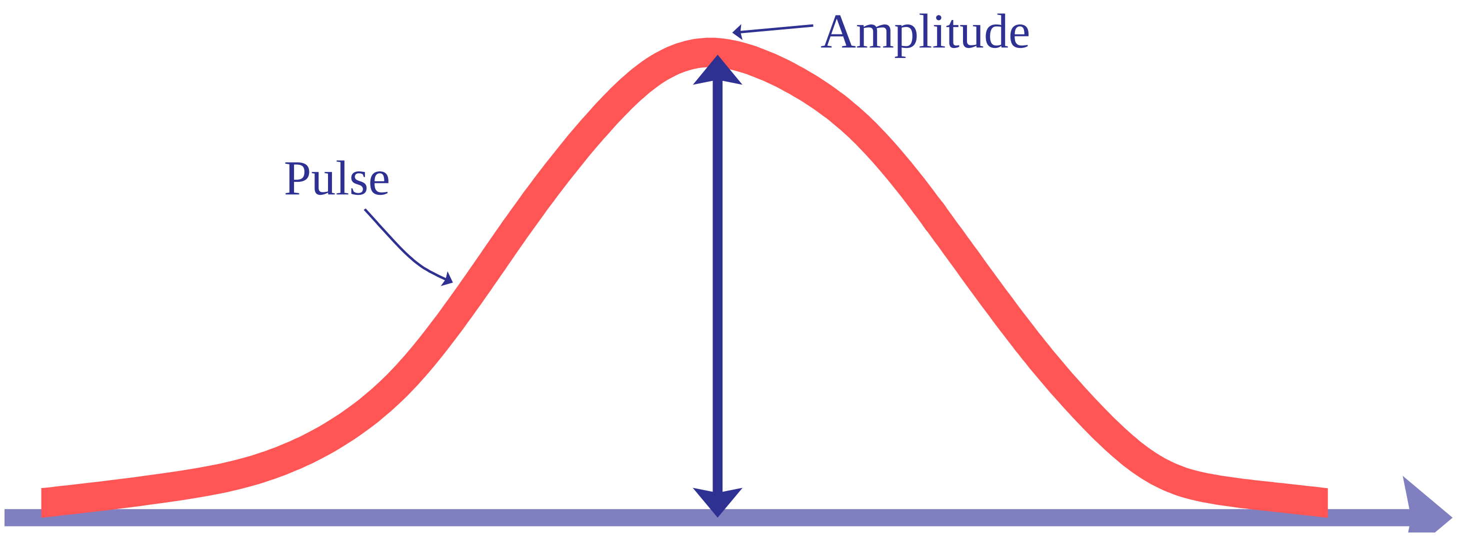
- Thus any N-dimensional signal can be represented geometrically as a point in the signal space spanned by the N orthonormal functions $\{\psi_j(t)\}$

- From the example we can represent the waveforms $s_1(t), \ldots, s_4(t)$ as

$$s_1 = [\sqrt{2}, 0, 0], s_2 = [0, \sqrt{2}, 0], s_3 = [0, -\sqrt{2}, 1], s_4 = [\sqrt{2}, 0, 1]$$

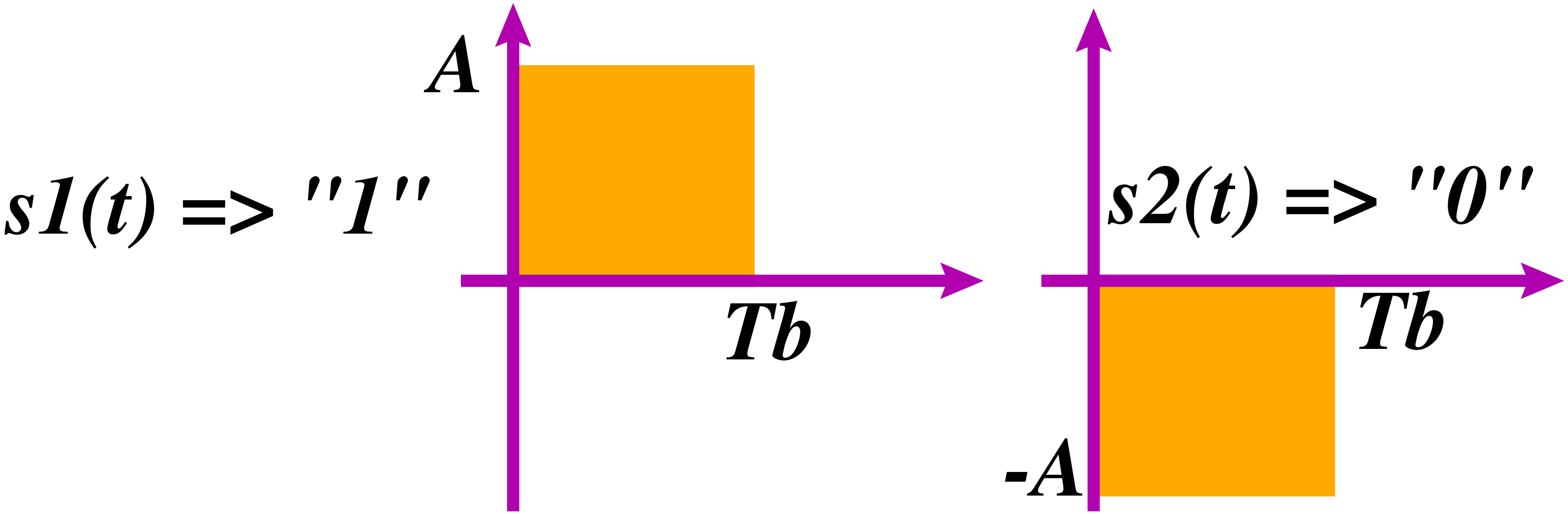
Pulse Amplitude Modulation (PAM)

- In *PAM* the *information* is conveyed by the *amplitude* of the transmitted (signal) pulse



Baseband PAM

- Binary PAM is the *simplest* digital modulation method
- A "1" bit may be represented by a pulse of amplitude A
- A "0" bit may be represented by a pulse of amplitude $-A$
- This is called *binary antipodal signalling*



Baseband PAM (cont.)

- The pulses are transmitted at a bit-rate of $R_b = 1/T_b$ bits/s where T_b is the bit interval (width of each pulse).

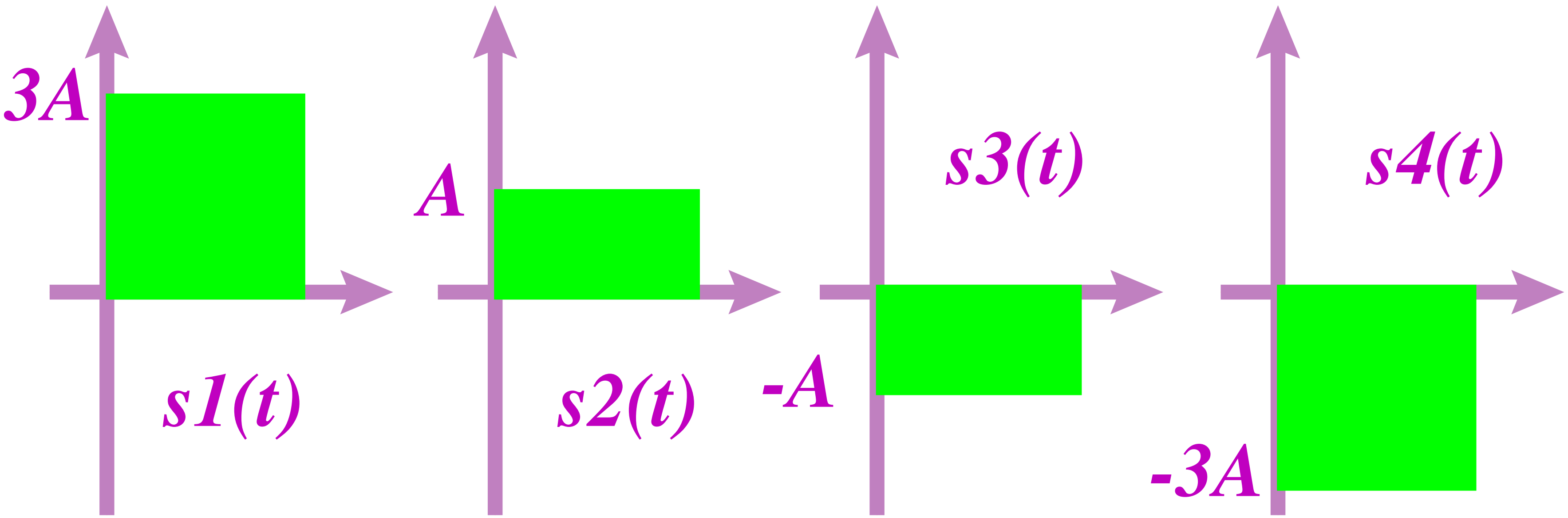
- We tend to show the pulse as *rectangular* ($\Rightarrow$ infinite bandwidth) but in practical systems they are more *rounded* ($\Rightarrow$ finite bandwidth)

- We can generalize PAM to M-ary pulse transmission ($M \geq 2$)

- In this case the binary information is subdivided into k-bit *blocks* where $M = 2^k$. Each k-bit block is referred to as a *symbol*.

- Each of the M k-bit symbols is represented by one of M *pulse amplitude values*.

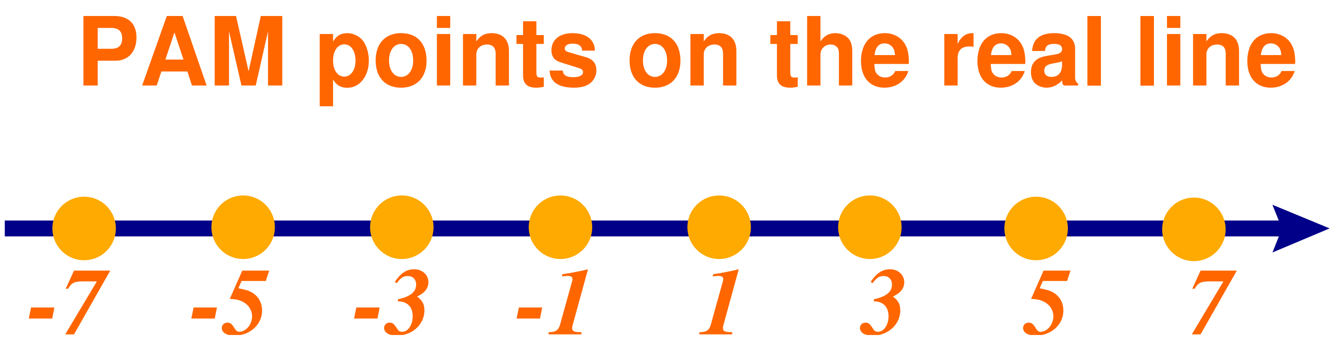
Baseband PAM (cont.)



- e.g., for $M = 4, k = 2$ bits per block, as we need 4 different amplitudes. The figure shows a rectangular pulse shape with amplitudes $\{3A, A, -A, -3A\}$ representing the bit blocks $\{01, 00, 10, 11\}$ respectively.

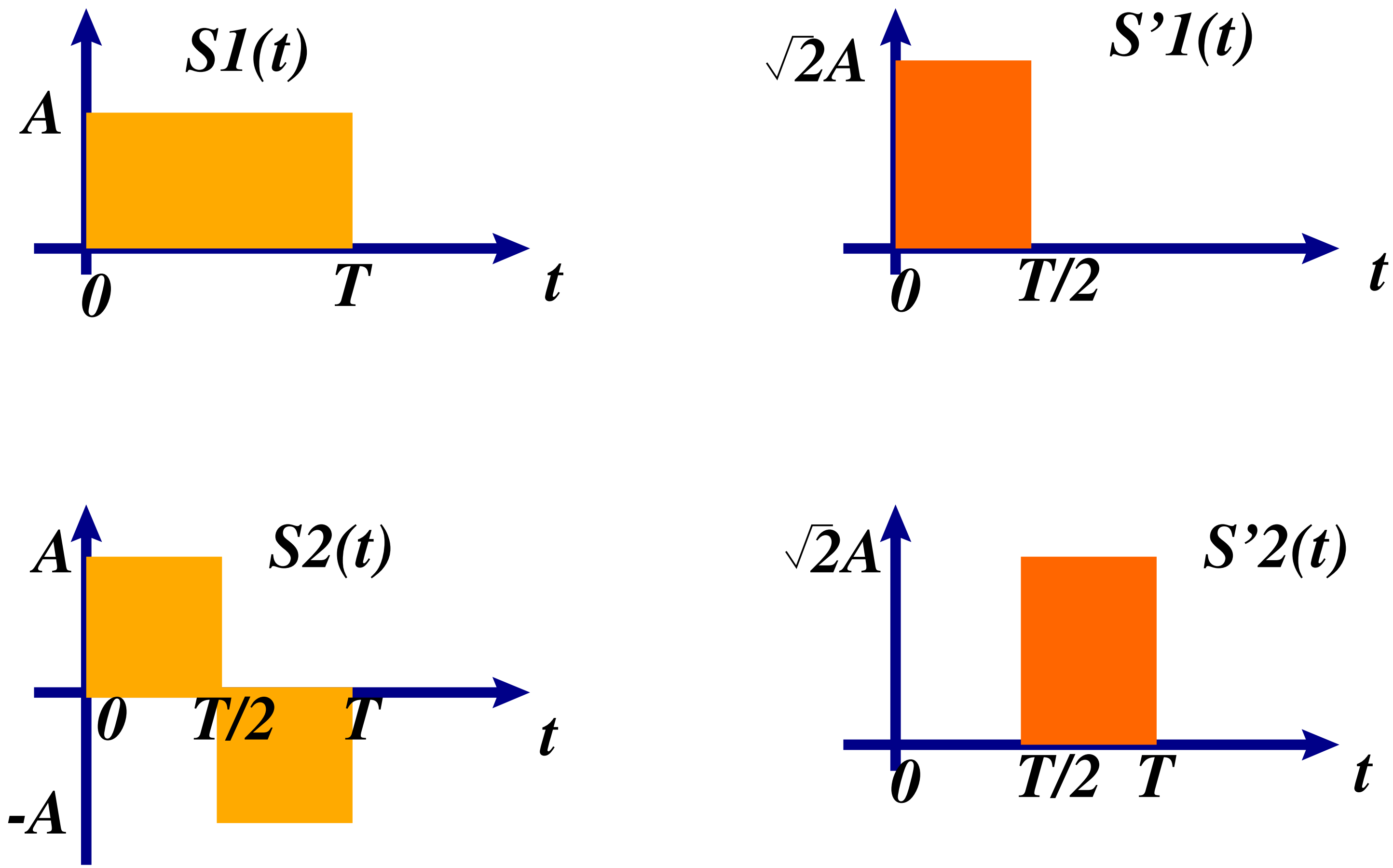
Two Dimensional Signals

- Recall that PAM signal waveforms are *one-dimensional*.
- That is, we could represent them as points on the real line, $\mathbb{R}$.



- We can represent signals of more than one dimension
- We begin by looking at *two-dimensional* signal waveforms

Orthogonal Two Dimensional Signals



Two Dimensional Signals (cont.)

- Recall that two signals are orthogonal over the interval $(0, T)$ if their *inner product*

$$\int_0^T s_1(t)s_2(t)dt = 0$$

- Can verify *orthogonality* for the previous (vertical) pairs of signals by observation

- Note that all of these signals have *identical energy*, e.g. energy for signal $s_2^{'}(t)$

$$\mathcal{E} = \int_0^T [s_2^{'}(t)]^2 dt = \int_{T/2}^T [\sqrt{2}A]^2 dt = 2A^2[t]_{T/2}^T = A^2T$$

Two Dimensional Signals (cont.)

- We could use either signal pair to transmit *binary information*
- One signal (in each pair) would represent a binary "1" and the other a binary "0"
- We can represent these signal waveforms as *signal vectors* in *two-dimensional space,* $\mathbb{R}^2$
- For example, choose the unit energy square wave functions as the basis functions $\psi_1(t)$ and $\psi_2(t)$

$$\psi_1(t) = \begin{cases} \sqrt{2/T}, & 0 \le t \le T/2 \\ 0, & \text{otherwise} \end{cases}$$

$$\psi_2(t) = \begin{cases} \sqrt{2/T}, & T/2 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

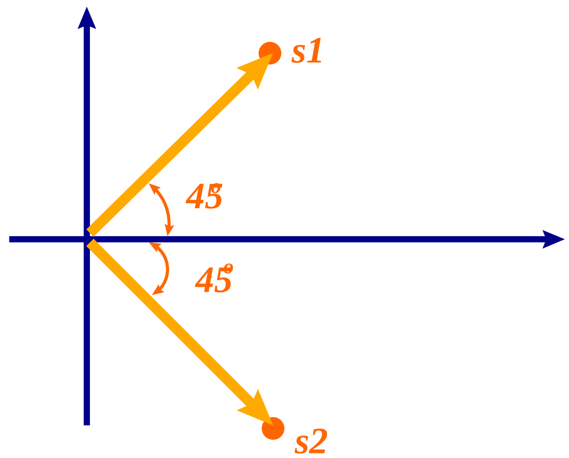
Two Dimensional Signal Waveforms (cont.)

- The waveforms $s_1(t)$ and $s_2(t)$ can be written as *linear combinations* of the basis functions

$$s_1(t) = s_{11}\psi_1(t) + s_{12}\psi_2(t)$$

$$\boldsymbol{s_1} = (s_{11}, s_{12}) = (A\sqrt{T/2}, A\sqrt{T/2})$$

- Similarly, $s_2(t) \equiv \boldsymbol{s_2} = (A\sqrt{T/2}, -A\sqrt{T/2})$



Two Dimensional Signal Waveforms (cont.)

- We can see that the previous two vectors are orthogonal in 2-D space

- Recall that their *lengths* give the *energy*

$$\mathcal{E}_1 = \|\boldsymbol{s_1}\|^2 = s_{11}^2 + s_{12}^2 = A^2 T$$

- The *euclidean distance* between the two signals is

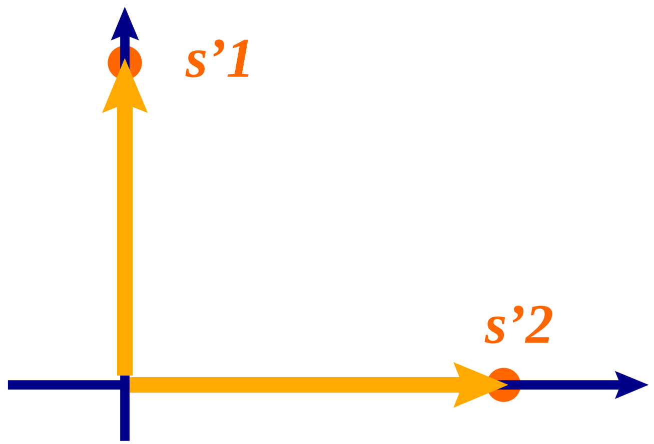
$$d_{12} = \sqrt{\|\boldsymbol{s_1} - \boldsymbol{s_2}\|^2} = \sqrt{\|(s_{11} - s_{21}, s_{12} - s_{22})\|^2} = \sqrt{\|(0, A\sqrt{2T})\|^2}$$

$$= A\sqrt{2T} = \sqrt{A^2 2T} = \sqrt{2\mathcal{E}}$$

Two Dimensional Signal Waveforms (cont.)

- Can similarly show that the other two waveforms are orthogonal and can be represented using the same basis functions $\psi_1(t)$ and $\psi_2(t)$

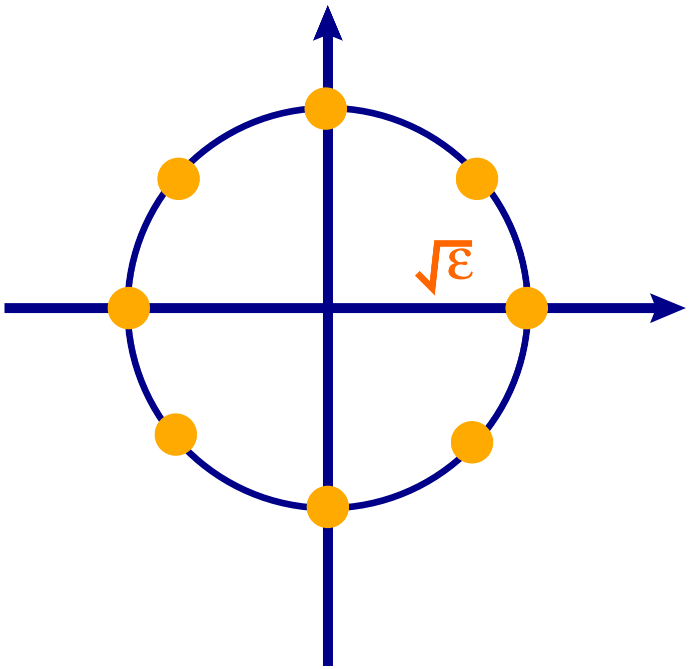
- Their *representative vectors* turn out to be a 45° *rotation* of the previous two vectors.



Representation of > 2 bits in 2-D

- Simply add more vector points

- The total number of points that we have, M, tells us how many bits k we can represent with each symbol, $M = 2^k$, e.g., $M = 8, k = 3$

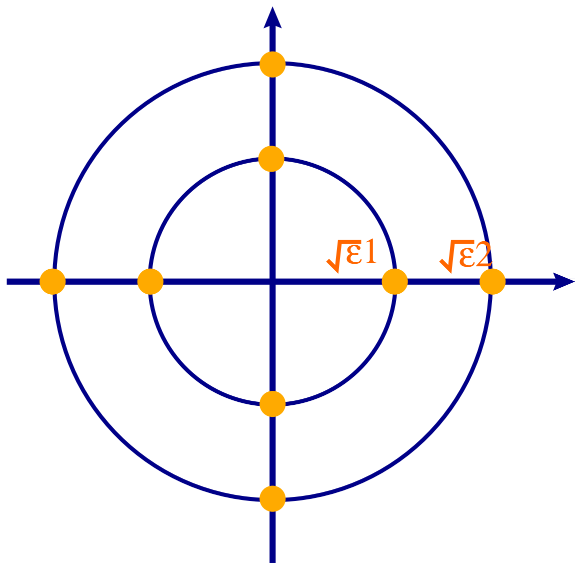


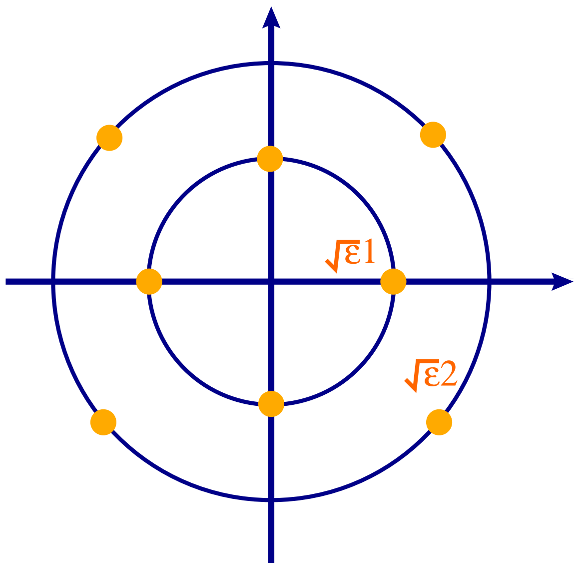
Representation of > 2 bits in 2-D (cont.)

- Note that the previous set of signals (vector representation) had *identical energies*

- Can also choose signal waveforms/points with *unequal energies*

- The constellation on the right gives an *advantage in noisy environments* (Can you tell why?)





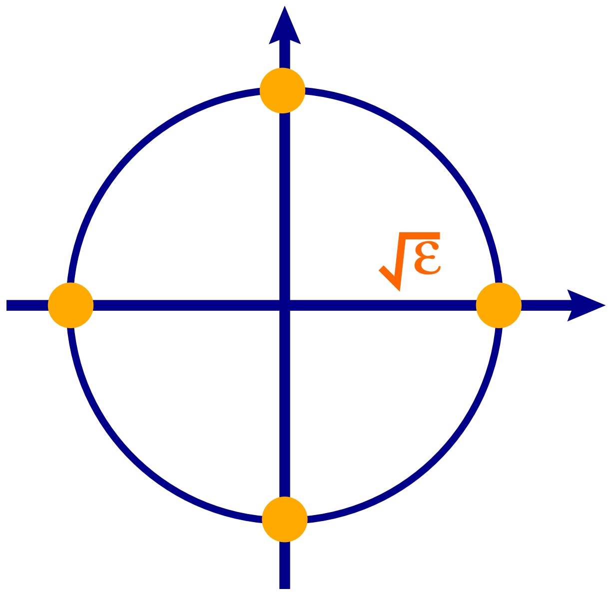
2-D Bandpass Signals

- Simply *multiply by a carrier*

$$u_m(t) = s_m(t)\cos 2\pi f_c t \qquad m = 1, 2, \ldots, M \quad 0 \le t \le T$$

- For $M = 4, k = 2$ and signal points with equal energies, we can have four *biorthogonal waveforms*

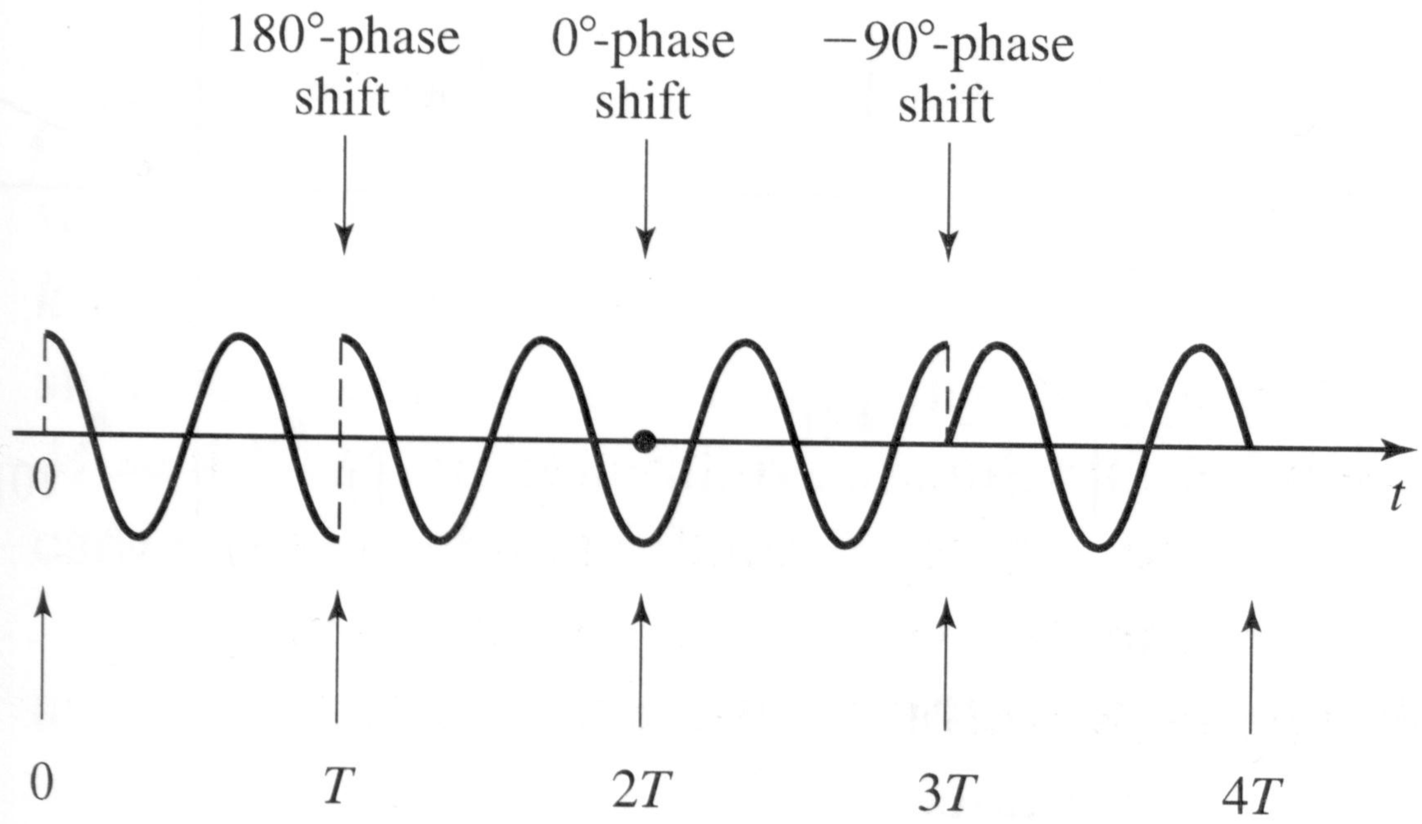
- These signal points/vectors are equivalent to *phasors*, where each is shifted by $\pi/2$ from each adjacent point/waveform



- For a rectangular pulse

$$u_m(t) = \sqrt{\frac{2\mathcal{E}_s}{T}}\cos\left(2\pi f_c t + \frac{2\pi m}{M}\right)$$

Carrier with Square Pulse



2-D Bandpass Signals

- This type of signalling is also referred to as *phase-shift keying (PSK)*

- Can also be written as

$$u_m(t) = g_T(t)A_{mc}\cos 2\pi f_c t - g_T(t)A_{ms}\sin 2\pi f_c t$$

where $g_T(t)$ is a square wave with amplitude $\sqrt{2\mathcal{E}_s/T}$ and width T, so that we are using a pair of *quadrature carriers*

- Note that *binary phase modulation* is identical to *binary PAM*

- A value of interest is the *minimum Euclidean distance* which plays an important role in determining *bit error rate performance* in the presence of AWGN.

Quadrature Amplitude Modulation (QAM)

- For MPSK, signals were constrained to have *equal energies*.
- The representative signal points therefore lay on a *circle* in 2-D space
- In *quadrature amplitude modulation (QAM)* we allow different energies.
- *QAM* can be considered as a combination of *digital amplitude modulation* and *digital phase modulation*

QAM

- Each bandpass waveform is represented according to a distinct amplitude/phase combination

$$u_{mn}(t) = A_m g_T(t) \cos(2\pi f_c t + \theta_n)$$