College of Computer Science
Northeastern University
Wireless Networks CS G250

Fall 2008
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Problem Set 1 - Solutions

1.(a)

$$s(t) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left((2k+1)\pi t\right)$$

If the signal is applied to an ideal low-pass filter with bandwidth of $15Hz$, then the output from the filter is (Figure 1):

$$s_{15}(t) = \frac{2}{\pi} \left(\sum_{k=0}^{14} \frac{(-1)^k}{2k+1} \cos\left((2k+1)\pi t\right) \right)$$

If the signal is applied to an ideal low-pass filter with bandwidth of $5Hz$, then the output from the filter is (Figure 2):

$$s_{5}(t) = \frac{2}{\pi} \left(\sum_{k=0}^{4} \frac{(-1)^k}{2k+1} \cos\left((2k+1)\pi t\right) \right)$$

If the signal is applied to an ideal low-pass filter with bandwidth of $5Hz$, then the output from the filter is (Figure 3):

$$s_{3}(t) = \frac{2}{\pi} \left(\sum_{k=0}^{2} \frac{(-1)^k}{2k+1} \cos\left((2k+1)\pi t\right) \right)$$

As the bandwidth increases, the signal approaches a square signal.

1.(b) If the signal is applied to a bandpass filter that passes frequencies from 5 to 9Hz, then the output from the filter is (Figure 4):

$$s_{5-9}(t) = \frac{2}{\pi} \left(\sum_{k=5}^{8} \frac{(-1)^k}{2k+1} \cos\left((2k+1)\pi t\right) \right)$$

The signal obtained by applying a bandpass filter doesn't resemble the original signal. The reason for this is that the most significant component was eliminated by the bandpass filter.

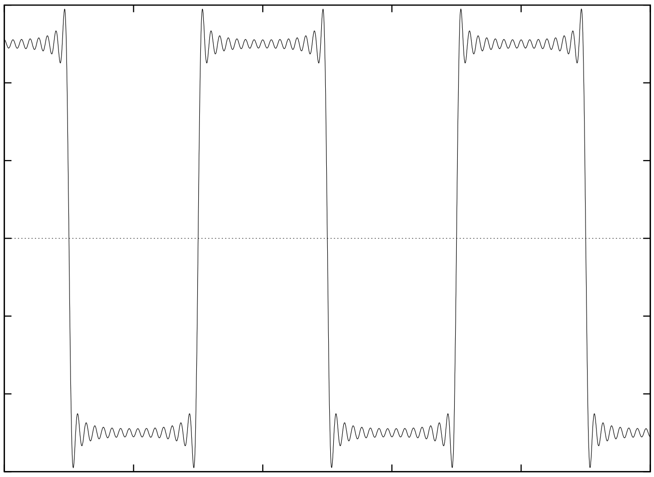

Figure 1: $s_{15}(t)$ signal applied to an ideal low-pass filter with bandwidth 15Hz

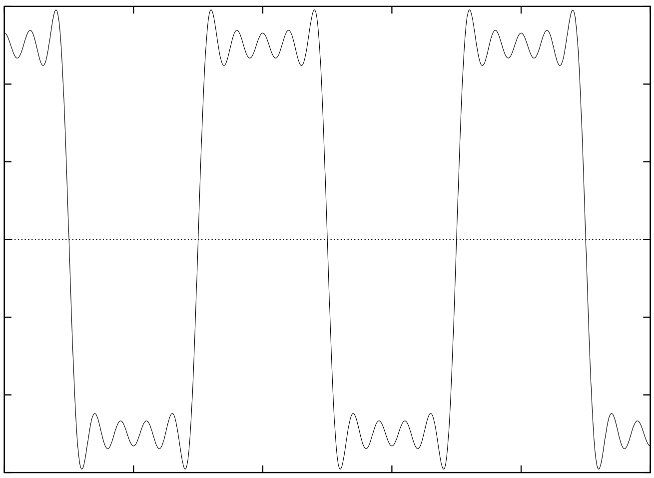

Figure 2: $s_5(t)$ signal applied to an ideal low-pass filter with bandwidth 5Hz

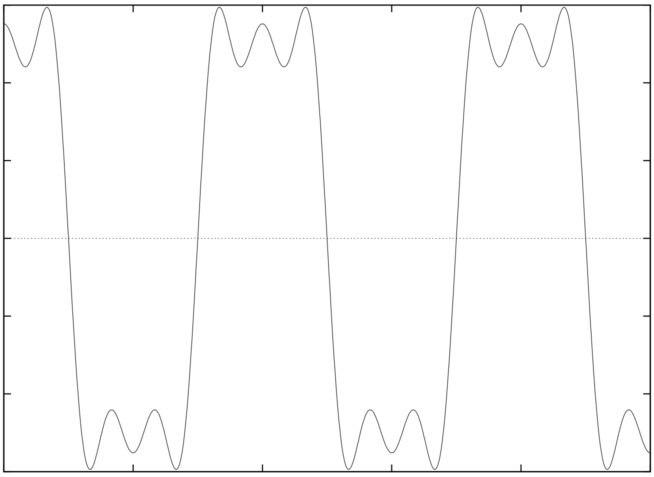

Figure 3: $s_3(t)$ signal applied to an ideal low-pass filter with bandwidth 3Hz

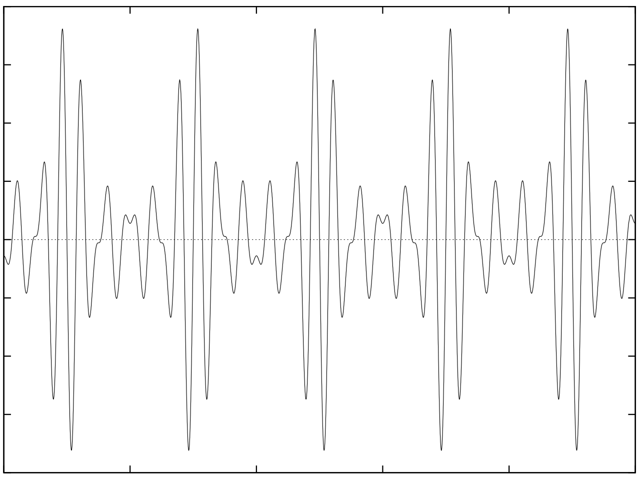

Figure 4: $s_{5-9}(t)$ signal applied to a bandpass filter that passes frequencies from 5 to 9Hz

2.(a)

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x(\tau)\, n(t-\tau)\, d\tau \\
Y(f) &= \int_{-\infty}^{\infty} y(t)\, e^{-2\pi i t f} dt \\
X(f) &= \int_{-\infty}^{\infty} x(\tau)\, e^{-2\pi i \tau f} d\tau \\
N(f) &= \int_{-\infty}^{\infty} n(\tau)\, e^{-2\pi i \tau f} d\tau
\end{aligned}$$

Now let's compute $Y(f)$

$$\begin{aligned}
Y(f) &= \int_{-\infty}^{\infty} y(t)\, e^{-2\pi i t f} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)\, n(t-\tau)\, d\tau \right) e^{-2\pi i t f} dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} n(t-\tau)\, e^{-2\pi i t f} dt \right) d\tau \\
t-\tau &= z \Rightarrow \\
t &= z+\tau \Rightarrow \\
dt &= dz \\
Y(f) &= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} n(z)\, e^{-2\pi i (z+\tau) f} dz \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} n(z)\, e^{-2\pi i z f} e^{-2\pi i \tau f} dz \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\, e^{-2\pi i \tau f} d\tau \int_{-\infty}^{\infty} n(z)\, e^{-2\pi i z f} dz \\
&= Y(f)\, N(f)
\end{aligned}$$

2.(b)

$$\begin{aligned}
x(t) &= e^{-at} \\
X(f) &= \int_0^{\infty} e^{-at} e^{-2\pi itf} dt \\
&= \left. \frac{e^{-2\pi itf-at}}{-a-2\pi if} \right|_0^{\infty} \\
&= 0 - \frac{e^0}{-a-2\pi if} \\
&= \frac{1}{a+2\pi if} \\
&= \frac{a-2\pi if}{a^2+4\pi^2 f^2} \\
\text{Phase} &= \arctan\left(\frac{\frac{-2\pi f}{a^2+4\pi^2 f^2}}{\frac{a}{a^2+4\pi^2 f^2}}\right) \\
&= \arctan\left(\frac{-2\pi f}{a}\right) \text{ (Figure 5)} \\
\text{Magnitude} &= \sqrt{\left(\frac{a}{a^2+4\pi^2 f^2}\right)^2 + \left(\frac{-2\pi f}{a^2+4\pi^2 f^2}\right)^2} \\
&= \frac{1}{\sqrt{a^2+4\pi^2 f^2}} \text{ (Figure 6)}
\end{aligned}$$

2.(c) While using sinusoidals when combining the signal with noise, the operation spills over both the sine and cosine terms. Using complex exponentials, the operation becomes cleaner.

3.(a) **Theorem 1** *Nyquist Sampling Theorem*
Let $x(t)$ be a signal whose frequency components are no greater than a certain frequency f_B. Then the original signal can be completely described from samples taken at a rate $f_s > 2f_B$

To sample a signal $x(t)$, we multiply it by a sampling signal $p(t)$, whose value is 1 when the sample is taken and 0 otherwise. So $x_s(t) = x(t)p(t)$ where $x_s(t)$ is the resulting sampled signal.
The sampling signal is periodic, so we can write it as

$$p(t) = \sum_{n=0}^{\infty} P_n e^{i2\pi nft}$$

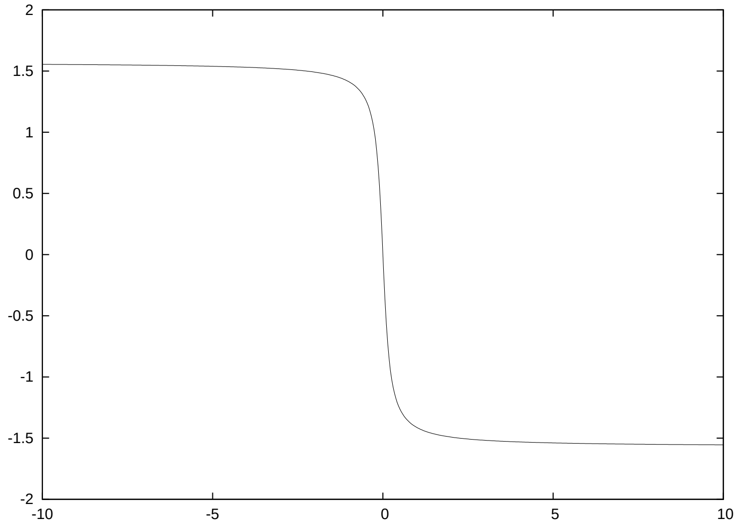

Figure 5: Phase $= \arctan\left(\frac{-2\pi f}{a}\right)$

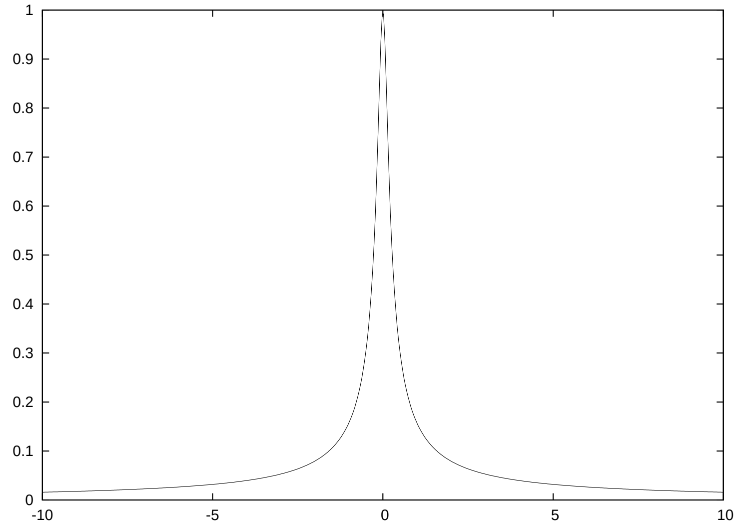

Figure 6: Magnitude $= \frac{1}{\sqrt{a^2+4\pi^2 f^2}}$

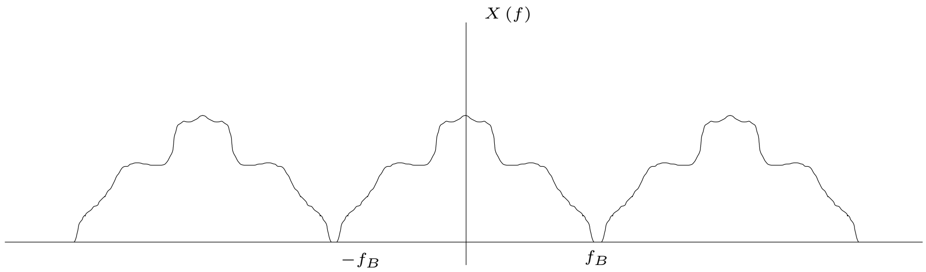

Figure 7: Sampled signal in the frequency domain. The original signal lies completely between $-f_B$ and f_B

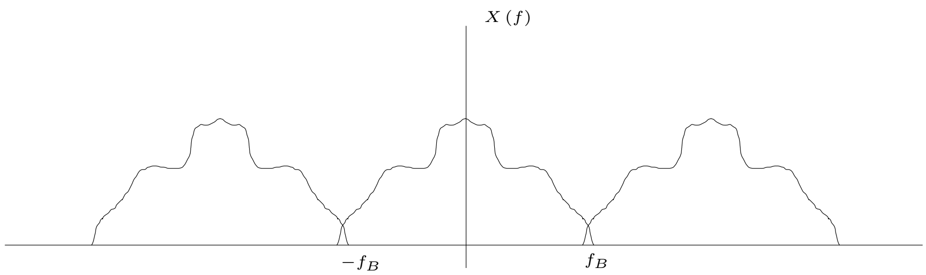

Figure 8: A signal sampled with a frequency $f < 2f_B$ will generate a graph in the frequency domain where the ends overlap

for some values of P_n.

What does the Fourier Transform of $x_s(t)$ look like? If we assume a sampling frequency of f_s

$$\begin{aligned}
X_s(f) &= \int_{-\infty}^{\infty} x_s(t) e^{-i2\pi f t} dt \\
&= \int_{-\infty}^{\infty} x(t) \sum_{n=0}^{\infty} P_n e^{i2\pi n f_s t} e^{-i2\pi f t} dt \\
&= \sum_{n=0}^{\infty} P_n \int_{-\infty}^{\infty} x(t) e^{i2\pi(n f_s - f)t} dt \\
&= \sum_{n=0}^{\infty} P_n \int_{-\infty}^{\infty} x(t) e^{-i2\pi(f - n f_s)t} dt \\
&= \sum_{n=0}^{\infty} P_n X(f - n f_s) \\
&= P_0 X(f) + P_1 X(f - f_s) + P_2 X(f - 2f_s) + \ldots
\end{aligned}$$

The Fourier Transform of the sample signal is made of "copies" of the Fourier Transform of the original signal located at f_s intervals away. Since the original signal has no frequency components over f_B, then its figure in the frequency domain looks like the central shape in Figure 7, which lies entirely between $-f_B$ and f_B. Then, when $f_s > 2f_B$, the copies will not overlap.

When $f_s \leq 2f_B$, the ends will overlap, making it impossible to retrace the original values of the signal. This is called undersampling and a signal obtained from such a processed is said to be *aliased*.

(b) By the definition of Laplace transform, and using

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right):$$

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} \cos\left(\frac{f_s}{2} t\right) e^{-2i\pi f t} dt \\
&= \frac{1}{2}\left(\int_{-\infty}^{\infty} e^{it(f_s/2 - 2\pi f)} + \int_{-\infty}^{\infty} e^{it(f_s/2 + 2\pi f)}\right) \\
&= \frac{1}{2}\left(\delta\left(f - f_s/4\pi\right) + \delta(f + f_s/4\pi)\right),
\end{aligned}$$

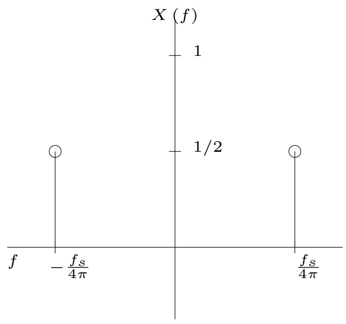

Figure 9: Problem 3.(b) $X(f)$ looks like two pulses of magnitude 1/2 at $\pm f_s/4\pi$

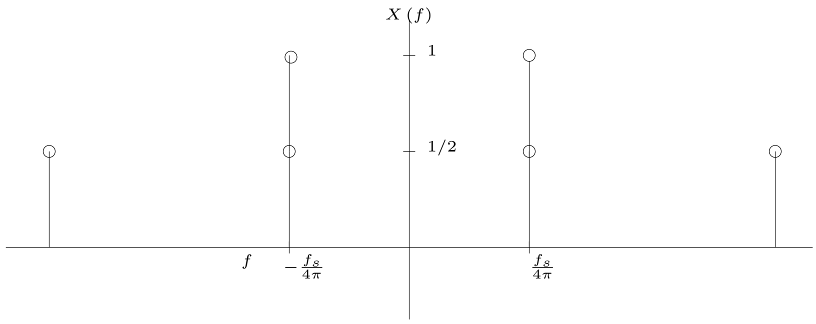

Figure 10: Problem 3.(b) $X_s(f)$ will have both its endpoints touching the adjacent copies' same endpoints

which looks like Figure 9.

The sampled signal (at frequency f_s) is:

$$\begin{aligned} X_s(f) &= \sum_n P_n X(f - nf_s/2\pi) \\ &= \frac{1}{2} \sum_n X(f - nf_s/2\pi) \end{aligned}$$

Note in $X_s(f)$ that at $f = \pm f_s/4\pi$, two copies of the delta function meet and are added together. The plot looks like Figure 10

After applying a bandpass filter to get rid of the higher frequencies, one ends up with a function that is different from the original. The result of the addition of both ends of the signals is an equation of the form $a + b = c$, where a, b are the endpoints of the curve and c is the result. There is an infinite number of values for a and b that satisfy the equation and it is thus impossible to recover the original.

4.(a) The system must transmit at 38.4 Kbps, with a symbol length of 8 bits. Thus, the rate is:

$$R = \frac{38.4 \times 10^3}{8} = 4800 \text{ symbols per second} = 4800\text{Hz}$$

Nyquist's Sampling Theorem states that the sampling rate must be at least twice as much as the frequency of the signal, therefore, the signal Bandwidth must be $B > 4800/2 = 2400$ Hz.

The signal-to-noise ratio is obtained with Shannon's formula:

$$\begin{aligned} C &= B \log_2 (1 + \text{SNR}) \\ \text{SNR} &= 2^{C/B} - 1 \\ &= 2^{16} - 1 \\ &= 65535 \\ \text{SNR}_{dB} &= 48.16 \end{aligned}$$

(b) The energy per bit is $E_b = ST_b$, where S is the signal power, and T_b is the time required to transmit one bit. Therefore $E_b = S/C$, where C is the capacity or data rate. Recall also that $N = N_0 B$ where B is the bandwidth of the transmission, and N_0 is the noise power density per Hz. So,

$$\begin{aligned} \frac{E_b}{N_0} &= \frac{SB}{NC} \\ S/N &= \frac{E_b}{N_0}\frac{C}{B} \\ C &= B \log_2(1 + S/N) \\ \text{SNR} &= 2^{C/B} - 1 \\ \frac{E_b}{N_0} &= \frac{B}{C}\left(2^{C/B} - 1\right) \\ &= \frac{2^r - 1}{r}, \text{ where } r = C/B \end{aligned}$$

The plot is shown in Figure 11

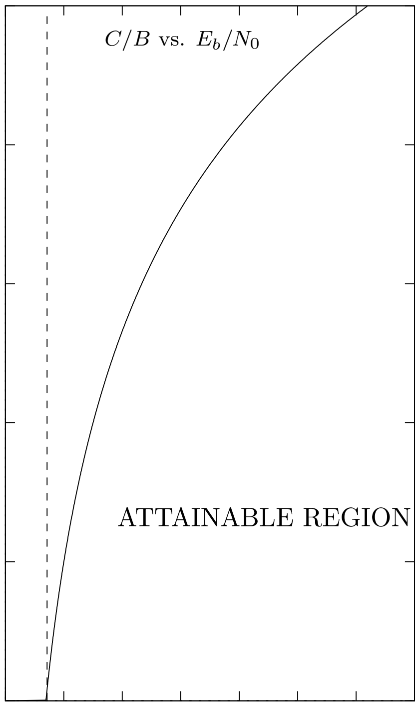

Figure 11: The E_b/N_0 plot in terms of C/B

To obtain the point at which the communication is 0, we need to obtain the value of $\frac{E_b}{N_0}$ when $r = 0$:

$$\begin{aligned}
\lim_{r\to 0} \frac{2^r - 1}{r} &= \lim_{r\to 0} \frac{e^{x \ln 2} - 1}{r} \\
&= \lim_{r\to 0} (\ln 2) e^{x \ln 2} \\
&= \ln 2 \\
\frac{E_b}{N_0}_{dB} &= -1.59
\end{aligned}$$

5.(a) Recall that

$$\frac{P_t}{P_r} = \frac{(4\pi fd)^2}{G_r G_t c^2}$$

Doubling either the frequency of the distance inserts a factor of two squared in the numerator and thus is equivalent. The attenuation of power in dB form for both cases is

$$\begin{aligned}
10 \log_{10} \frac{P_t}{P_r} &= 10 \log_{10} \frac{(4\pi fd)^2}{G_r G_t c^2} \\
10 \log_{10} \frac{P_t}{P_r} &= 10 \log_{10} \left(4 \frac{(4\pi fd)^2}{G_r G_t c^2} \right)
\end{aligned}$$

(b) Let's look at Figure 12. By applying Pythagoras' Theorem, we get the following:

$$\begin{aligned}
(h + R)^2 &= d^2 + R^2 \\
h^2 + 2hR + R^2 &= d^2 + R^2 \\
d^2 &= h^2 + 2hR \\
R &= \frac{d^2 - h^2}{2h}
\end{aligned}$$

Now let's remember that d is in km and h is in meters.

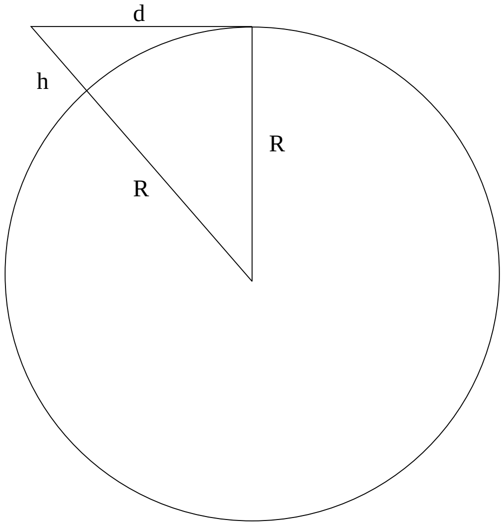

Figure 12: LOS

$$
\begin{aligned}
R &= \frac{d^2 - \left(\frac{h}{1000}\right)^2}{2\left(\frac{h}{1000}\right)} \\
&= \frac{\left(4\sqrt{h}\right)^2 - \left(\frac{h}{1000}\right)^2}{2\left(\frac{h}{1000}\right)} \\
&= \frac{16h - \frac{h^2}{10^6}}{\frac{2h}{10^3}} \\
&= \frac{\frac{16\times 10^6 h - h^2}{10^6}}{\frac{2h}{10^3}} \\
&= \frac{16 \times 10^6 - h}{2 \times 10^3} \\
&= \frac{16 \times 10^6}{2 \times 10^3} - \frac{h}{2 \times 10^3} \\
&= 8 \times 10^3 - \frac{h}{2 \times 10^3} \\
R &\approx 8km
\end{aligned}
$$

(c) Plugging the values given into the power ratio formula gives:

$$\begin{aligned} P_r &= \frac{G^2 c^2 P_t}{16\pi^2 f^2 d^2} \\ &= \frac{100 \cdot 9 \cdot 10^{16} \cdot 0.5}{16\pi^2 \cdot 9 \cdot 10^{16} \cdot 25 \cdot 10^6} \\ &= \frac{2}{16\pi^2 \cdot 10^6} \\ &= 1.27 \times 10^{-8}\text{W} \end{aligned}$$