College of Computer Science
Northeastern University
Wireless Networks CS G250

Lecture 3
September 17 2008
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Shannon's Theorem, Antennas

Handouts: Problem Set 1
You can also download PS1 from
`http://www.ccs.neu.edu/course/csg250/ps1.pdf`
Lecture Outline:

- Shannon's Theorem
- Antennas

1 Shannon's Theorem

1.1 Description

Theorem 1 *Shannon's Theorem (in words): given a channel with noise there exists a non-zero capacity of data rate achievable with arbitrarily low error.*

$$\begin{aligned} \mathtt{C} &= \mathtt{B}\log_2\frac{\mathtt{P}+\mathtt{N}}{\mathtt{N}}, \ \mathtt{where} \\ \mathtt{C} &= \mathtt{capacity}, \\ \mathtt{B} &= \mathtt{bandwidth}, \\ \mathtt{P} &= \mathtt{power}, \\ \mathtt{N} &= \mathtt{noise} \end{aligned}$$

The most used model for noise is Additive White Gaussian Noise (AWGN). Using this model, every value of `y(t)`is shifted randomly with a normal probability distribution with respect to the original `x(t)`, as shown in Figure 1. The parameter `n`is the variance of the normal distribution. As a result, the signal `y(t)`will be on average the same as `x(t)`.

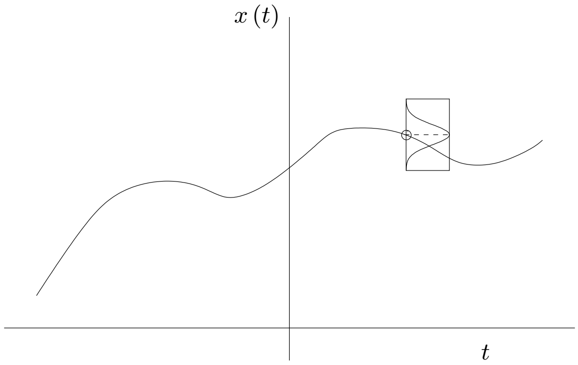

Figure 1: Using AWGN as a noise model, every point in `x(t)`will be shifted up or down randomly, with certain probability. The peak of the normal distribution lies exactly at `x(t)`

$$
\begin{aligned}
\mathtt{y(t)} &= \mathtt{x(t)} + \mathtt{N}(0, \mathtt{n}), \ \mathtt{where} \\
\mathtt{n} &= \ \mathtt{variance,} \\
\mathtt{N} &= \mathtt{The\ Normal\ distribution}
\end{aligned}
$$

Once the original signal `y(t)`, which includes noise, has been received, the next step is reconstructing `x(t)`. It is possible that 2 or more different signals match the received `y(t)`, so the problem reduces to finding enough separation between the possible values of `x(t)`such that no confusion may occur.

Suppose the signal `x(t)`is sampled for an interval of time `T`and the signal is sampled at `2B` samples/sec, giving us a vector $\langle x_1, x_2, \ldots, x_{2\mathtt{B}\ \mathtt{T}} \rangle$, and also a noise vector $\langle n_1, n_2, \ldots, n_{2\mathtt{B}\ \mathtt{T}} \rangle$. Recall the formula for the total energy of a signal (per unit resistance) `x(t)`, so the total energy of the signal and the noise function are:

$$
\begin{aligned}
E &= \int_{-\infty}^{\infty} |\mathtt{x(t)}|^2 \, dt = \frac{1}{2\mathtt{B}} \sum_{k=1}^{2\mathtt{B}\ \mathtt{T}} {x_i}^2 \\
E &= \mathtt{P}\ \mathtt{T} \\
\frac{1}{2\mathtt{B}} \sum_{k=1}^{2\mathtt{B}\ \mathtt{T}} {x_i}^2 &= \mathtt{P}\ \mathtt{T} \\
\frac{1}{2\mathtt{B}} \sum_{k=1}^{2\mathtt{B}\ \mathtt{T}} {n_i}^2 &= \mathtt{N}\ \mathtt{T},
\end{aligned}
$$

where P and N is the signal and noise powers respectively, B is the signal Bandwidth and T is the sampling period.

The $\langle x_1, x_2, \ldots, x_{2\mathtt{B}\ \mathtt{T}} \rangle$ vector is 2B T-dimensional, and its length is $\sqrt{2\mathtt{BP}\ \mathtt{T}}$. This is a possible signal for the sender, which will be jittered by the energy of the noise inside a 2B T-dimensional ball of radius $\sqrt{2\mathtt{B}\ \mathtt{TN}}$. Now the question has been reduced to how many "noise-sized" balls fit inside the "signal-sized" ball? This is illustrated in Figure 2

$$
\begin{aligned}
\frac{Volume\ of\ big\ ball}{Volume\ of\ small\ ball} &= \frac{cR^d}{cr^d} \\
&= \left(\frac{R}{r}\right)^d \\
&= \left(\sqrt{\frac{2\mathtt{B}\ \mathtt{T}(\mathtt{P}+\mathtt{N})}{2\mathtt{B}\ \mathtt{TN}}}\right)^{2\mathtt{B}\ \mathtt{T}} \\
&= \left(\frac{\mathtt{P}+\mathtt{N}}{\mathtt{N}}\right)^{\mathtt{B}\ \mathtt{T}}
\end{aligned}
$$

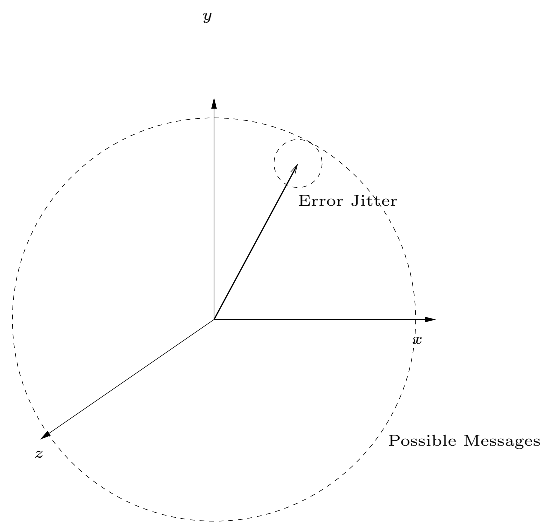

Figure 2: A possible signal `x(t)`may get jittered around due to noise by a certain amount limited to the noise power

$$
\begin{aligned}
\#bits &= \mathtt{B}\ \mathtt{T} \log_2 \frac{\mathtt{P}+\mathtt{N}}{\mathtt{N}} \\
\mathtt{C} &= \mathtt{B} \log_2 \frac{\mathtt{P}+\mathtt{N}}{\mathtt{N}} \\
&= \mathtt{B} \log_2 \left(1 + \frac{\mathtt{P}}{\mathtt{N}}\right), \ \ \mathtt{where} \\
\frac{\mathtt{P}}{\mathtt{N}} &= SNR
\end{aligned}
$$

We define the SNR = signal to noise ratio (measured in decibels) as the ratio of the power in a signal to the power contained in the noise present at a particular point in the transmission. Typically, the ratio is measured at the receiver.

$$SNR_{dB} = 10 \log_{10} \frac{signal\ power}{noise\ power}$$

The formula for computing the number of decibels of value x is $X = 10log_{10}x$, where X is the value in decibels.

1.2 Implications of Shannon's Theorem

$$\mathtt{C} \;=\; \mathtt{B}\log_2 \frac{\mathtt{P}+\mathtt{N}}{\mathtt{N}}$$

Shannon's Theorem is universally applicable (not only to wireless).

If we desire to increase the capacity in a transmission, then one may increase the Bandwidth and/or the transmission power. Two questions arise:

- Can B be increased arbitrarily?

  No, because of:

  - regulatory constraints
  - semiconductor constraints (silicon)
  - bandwidth noise increases with bandwidth

$$\begin{aligned}
\mathtt{N} &= N_0\mathtt{B},\ \texttt{where} \\
N_0 &= \mathtt{kT} \\
N_0 &= \texttt{ Noise Power Density} \\
\mathtt{k} &= \texttt{ Boltzmann constant} \\
\mathtt{T} &= \texttt{ Temperature (in Kelvin)} \\
\texttt{Therefore,} & \\
\mathtt{C} &= \mathtt{B}\log_2\left(1+\frac{\mathtt{P}}{N_0\mathtt{B}}\right)
\end{aligned}$$

- Can P be increased arbitrarily?

  No, because of:

  - it's harmful
  - cost
  - regulatory reasons (you don't want to bleed into different spectrums)

There is a parameter that is very convenient to use for determining digital data rates and error rates. This parameter is called the ratio of signal energy per bit to noise power density per Hertz.

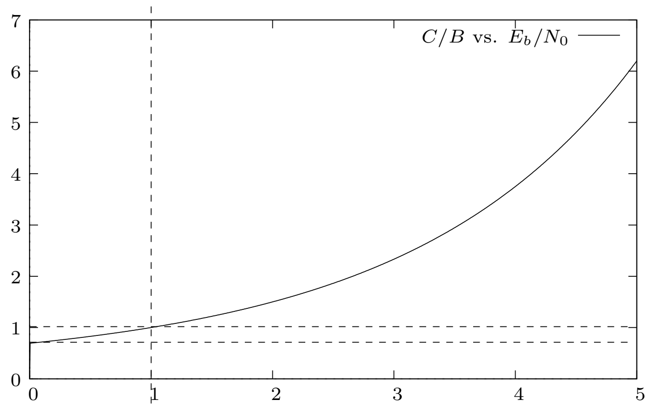

Figure 3: $\frac{C}{B}$ as a function of $\frac{E_b}{N_0}$

$$
\begin{aligned}
\frac{E_b}{N_0} &= \texttt{bit energy to noise density ratio} \\
E_b &= \frac{\texttt{P}}{\texttt{C}},\ \texttt{where} \\
E_b &= \texttt{ energy per bit} \\
N_0 &= \texttt{noise density}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\texttt{C} &= \texttt{B}\log_2\left(1+\frac{\texttt{P}}{N_0\texttt{B}}\right) \\
\texttt{C} &= \texttt{B}\log_2\left(1+\frac{E_b}{N_0}\frac{\texttt{C}}{\texttt{B}}\right) \\
\frac{\texttt{C}}{\texttt{B}} &= \log_2\left(1+\frac{E_b}{N_0}\frac{\texttt{C}}{\texttt{B}}\right),\ \texttt{where} \\
\frac{\texttt{C}}{\texttt{B}} &= \texttt{spectral efficiency, measured in } \frac{bits}{Hz}
\end{aligned}
$$

For a graphical representation of equation

$$\frac{\texttt{C}}{\texttt{B}} = \log_2\left(1+\frac{E_b}{N_0}\frac{\texttt{C}}{\texttt{B}}\right),\ \texttt{where}$$

please take a look at Figure 3.

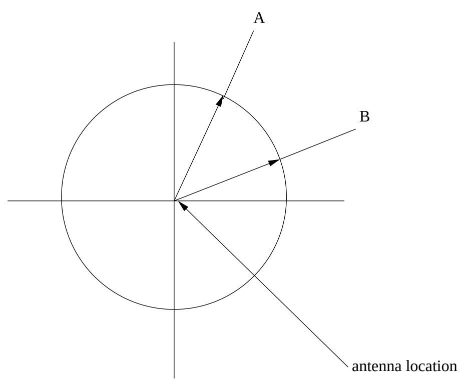

Figure 4: Omnidirectional antenna

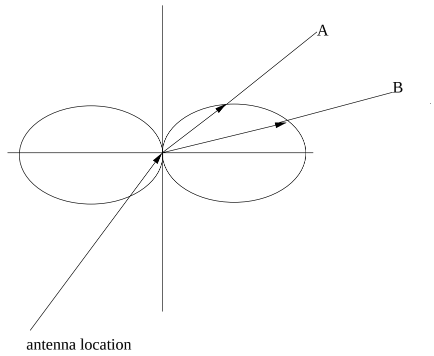

Figure 5: Directional antenna

2 Antennas

An antenna can be defined as an electrical conductor or system of conductors used either for radiating electromagnetic energy of for collecting electromagnetic energy. Usually the same antenna can be used for both transmission and reception. An antenna will radiate power in all directions, but it typically does not perform equally well in all directions.
A common way of characterizing the performance of an antenna is the radiation pattern.
Idealized radiation patterns:

- omnidirectional/isotropic (Figure 4)
- directional (Figure 5)

Antenna gain is a measure of the directionality of an antenna.

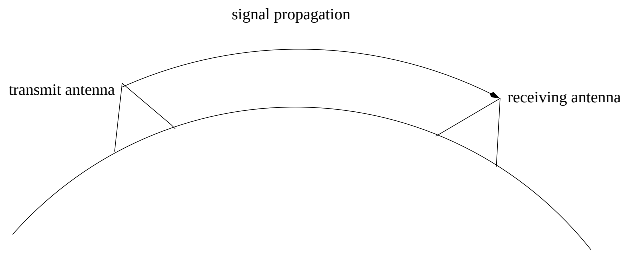

Figure 6: Ground wave propagation (below 2MHz)

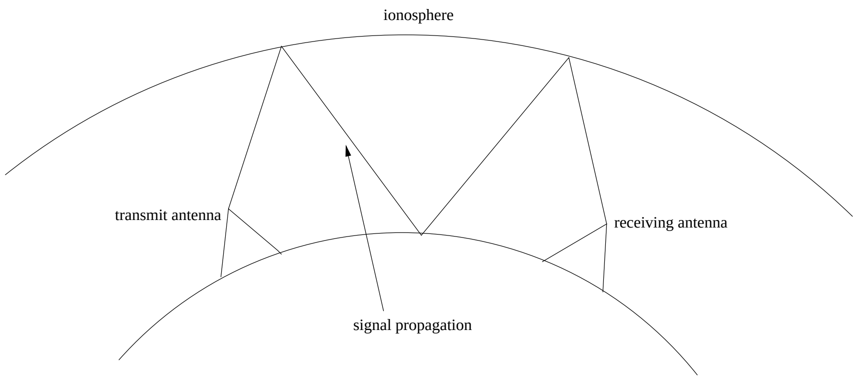

Figure 7: Sky wave propagation (2 to 30 MHz)

$$\begin{aligned} \texttt{G} &= \frac{4\pi A_e}{\lambda^2}, \texttt{ where} \\ \texttt{G} &= \texttt{ ratio of the antenna's power to that of an equivalent isotropic antenna} \\ \lambda &= \texttt{wave length} \\ A_e &= \texttt{Effective Area of Antenna} \end{aligned}$$

Propagation types:

- ground-wave (below 2MHz)(Figure 6)
- sky-wave (2 to 30MHz)(Figure 7)
- line of sight (LOS; above 30MHz)(Figure 8)

The optical line of sight (with no obstacles) is:

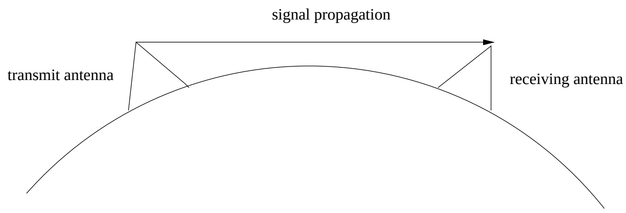

Figure 8: LOS propagation (above 30MHz)

$$\begin{aligned} \texttt{d} &= 3.57\sqrt{\texttt{h}},\ \texttt{where} \\ \texttt{d} &= \texttt{distance between an antenna and the horizon (in Kms)} \\ \texttt{h} &= \texttt{height of the antenna (in meters)} \end{aligned}$$

The radio line of sight to the horizon is:

$$\begin{aligned} \texttt{d} &= 3.57\sqrt{\texttt{hk}},\ \texttt{where} \\ \texttt{k} &= \frac{4}{3},\texttt{an adjustment factor for the refraction} \end{aligned}$$

The maximum distance between two antennas for LOS propagation is:

$$\begin{aligned} \texttt{d} &= 3.57\sqrt{\texttt{k}}(\sqrt{h_1}+\sqrt{h_2}),\ \texttt{where} \\ h_1 &= \texttt{height of the first antenna measured in meters} \\ h_2 &= \texttt{height of the second antenna measured in meters} \end{aligned}$$

The signal disperses with distance. A receiving antenna will receive less signal power the farther it is from the transmitting antenna. This form of attenuation is called free space loss, which is defined as:

$$\begin{aligned}
\frac{P_t}{P_r} &= \frac{(4\pi \mathtt{d})^2}{\lambda^2} \\
&= \frac{(4\pi \mathtt{d})^2}{\lambda^2 G_r G_t}, \ \texttt{where} \\
P_t &= \texttt{signal power of the transmitting antenna} \\
P_r &= \texttt{signal power of the receiving antenna} \\
G_t &= \texttt{gain of the transmitting antenna} \\
G_r &= \texttt{gain of the receiving antenna}
\end{aligned}$$