Lecture 4: September 27, 2018 [1]

Instructors: Adrienne Slaughter, Tamara Bonaci

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this class only with the permission of the Instructor.*

Sets and Matrices

Readings for this week:
Rosen, Chapter 2.1, 2.2, 2.5, 2.6
Sets, Set Operations, Cardinality of Sets, Matrices

4.1   Overview

1. Sets
2. Set Operations
3. Cardinality of Sets
4. DeMorgan's Law for Sets
5. Matrices

4.2   Introduction

4.3   Situating Problem Introduction

4.4   Sets

A **set** is a group of objects, usually with some relationship or similar property. The objects in the set are called **elements** or **members** of the set. A set **contains** its elements.

We use the symbol $\in$ to indicate that an element is or is not in a set:

$$x \in A: x \text{ is in set } A$$

$$x \notin A: x \text{ is not in set } A$$

A set is described by either listing out the elements of the set in braces, or using **set builder** notation.

- **Example:** The set V of vowels in English is $V = \{a, e, i, o, u\}$
- **Example:** The set O of all odd positive integers less than 10 is: $O = \{x \mid x$ is an odd positive integer less than $10\}$
- We use uppercase letters to denote a set.

Special Sets

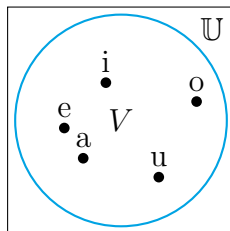$\mathbb{N}, \mathbb{Z}, \mathbb{R}$ are reserved to represent special sets:

- $\mathbb{N}$: the natural numbers $\{1, 2, ...\}$
- $\mathbb{Z}$: the set of integers $\{..., -2, -1, 0, 1, 2, ...\}$
- $\mathbb{Z}^+$: the set of non-negative integers $\underline{\{0, 1, 2, ...\}}$
- $\mathbb{R}$: the set of real numbers.
- $\emptyset$: the empty set (no elements)

Set Equality

Two sets are equal if and only if they contain the same elements.

Venn Diagrams

A ***Venn diagram*** is a graphical representation of a set.

The rectangle represents U, the ***universal set*** . The universal set is the set that contains all objects under consideration. In this example, U is the set of all letters, and the set V is the set of vowels. Specific elements are represented by a point (labeled or not).

Subset

The set A is a ***subset*** of B if and only if every element of A is also an element of the set B. We use the notation: $A \subseteq B$
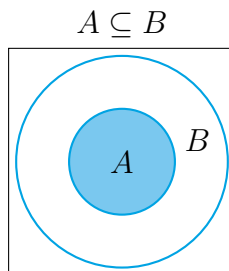
Another way to think about a subset:

$$\text{If } A \subseteq B, \text{ then } \forall x (x \in A \rightarrow x \in B)$$

The null set $\emptyset$ is a subset of every set: To see that $\emptyset \subseteq S$, see that every element in $\emptyset$ is in S .

Further, every set is a subset of itself.

Proper Subset

When A is a subset of B and $A \neq B$, we say that A is a ***proper subset*** of B, and write it:

$$A \subset B$$

$$A = \{1, 2, 3\}$$
$$B = \{x : 0 < x \leq 4\}$$
$$\Rightarrow A \subset B$$

Use subsets to prove equality of sets

A and B have the same elements if each set is a subset of the other.

For example:

$$A : \{a, b, c, d\}$$
$$B : \{a, b, c, d\}$$
$$A \subseteq B \text{ (all elements in } A \text{ are in } B)$$
$$B \subseteq A \text{ (all elements in } B \text{ are in } A)$$
$$\Rightarrow A = B$$

Cardinality

Let S be a set. If there are n distinct elements in S (and n is an integer greater than or equal to 0), S is a ***finite set***, and n is the ***cardinality*** or S.

The cardinality of S is written $|S|$

Cardinality: Examples

Example: Let A be the set of odd positive integers less than 10. What's $|A|$?

Answer: 5

Example: Let S be the set of letters in the alphabet. What's $|S|$?

Answer: **26**

Example: What's $|\emptyset|$?

Answer: **0**

Power Set

Let S be a set. The ***power set*** of S is the set of all subsets of the set S.

The power set of S is written $\mathcal{P}(S)$.

Let S be the set $\{0, 1, 2\}$.

$\mathcal{P}(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$

Note: The empty set and S (the set itself) are members of the power set.

Power Set: Note

Example: What is the power set of the empty set?

Answer: $\mathcal{P}(\emptyset) = \{\emptyset\}$

Example: What is the power set of the set $\{\emptyset\}$?

Answer: $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$

n-tuples

Sets are unordered, but we usually care about the ordering of elements.

For example, we may have a bunch of words, but it would be easier to search them if they're sorted, or put in a particular order.

Ordered n-tuple

The ***ordered n-tuple*** $(a_1, a_2, \ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element, and a_n as its nth element.

Ordered n-tuples are equal if and only if each corresponding pair of their elements are equal:

$$(a_1, a_2, \ldots a_n) = (b_1, b_2, \ldots b_n) \text{ if and only if:}$$
$$a_i = b_i \text{ for } i = 1, 2, \ldots n.$$

Ordered Pairs

A 2-tuple is called a ***ordered pair***.

The ordered pair (a, b) equals the ordered pair (c, d) if and only if $a = c$ and $b = d$.

(a, b) only equals (b, a) if $a = b$.

Cartesian Products

The ***Cartesian product*** of sets A and B (denoted $A \times B$) is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$A \times B = \{(a, b) | a \in A \wedge b \in B\}$

Example: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Answer: $A \times B = $ _____ $\{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$ _____

Cartesian Products of multiple sets

The **_Cartesian product_** of sets A_1, A_2, ... A_n denoted $A_1 \times A_2 \ldots \times A_n$ is the set of n-tuples $(a_1, a_2, \ldots a_n)$ where $a_i \in A_i$ for $i = 1, 2, \ldots n$.

$$A_1 \times A_2 \ldots \times A_n = \{(a_1, a_2, \ldots a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\} \tag{4.1}$$
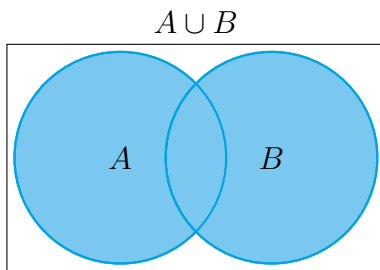
$$A : \{a, b, c\}$$
$$B : \{1, 2, 3\}$$
$$C : \{\text{blue, red, green}\}$$
$$\Rightarrow A \times B \times C = \{(a, 1, \text{blue}), (a, 1, \text{red}), (a, 1, \text{green}), (a, 2, \text{blue}) \ldots\}$$

The Cartesian product $A \times B \times C$ consists of all ordered triples (a, b, c), where $a \in A, b \in B, c \in C$.

Union

Let A and B be sets. The **_union_** of the sets A and B, denoted $A \cup B$ is the set that contains the elements in either A or in B, or in both.
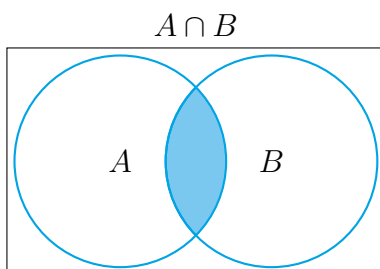
$A \cup B$

$$A \cup B = \{x | x \in A \lor x \in B\} \tag{4.2}$$

Example: $\{1, 2, 3, 4\} \cup \{7, 8, 9\} = ?$

Answer: _____ $\{1, 2, 3, 4, 7, 8, 9\}$ _____

Intersection

Let A and B be sets. The **_intersection_** of the sets A and B, denoted $A \cap B$ is the set that contains the elements in both A and B.
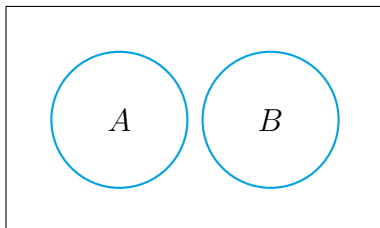
$A \cap B$

$$A \cap B = \{x | x \in A \land x \in B\} \tag{4.3}$$

Example: $\{1, 2, 3, 4\} \cap \{x : x \in N\} = ?$

Answer: _____ $\{1, 2, 3, 4\}$ _____

Disjoint

Let A and B be sets. The two sets are ***disjoint*** if their intersection is the empty set.

Cardinality of sets

Inclusion-Exclusion Principle

We frequently want too know how many items are in groups.

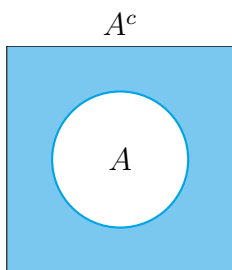Note that $|A| + |B|$ counts all the elements in **only** A once, and all the elements in **only** B once. BUT, it counts the elements in A **and** B TWICE.

Therefore, we need to subtract the number of elements that are in A and B:

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{4.4}$$

Complement of sets

The ***complement*** of a set A, denoted A^c is the set of elements that belong to U but which do not belong to A: t

A^c
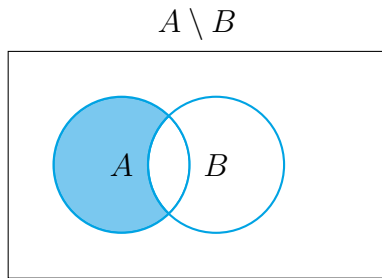

$$A^c = \{x : x \in U, x \notin A\} \tag{4.5}$$

$$U : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$
$$A : \{2, 5, 8\}$$
$$A^C = \underline{\quad \{1, 3, 4, 6, 7, 9, 10\} \quad}$$

Difference of sets

The ***relative complement*** or ***difference*** of a set B with respect to A, denoted $A \setminus B$ (said A minus B) is the set of elements that belong to A but which do not belong to B.
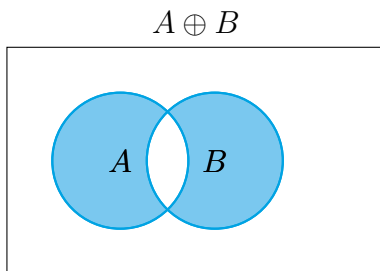
$$A \setminus B$$

$$A \setminus B = \{x : x \in A, x \notin B\} \qquad (4.6)$$

Example: $\{5, 6, 7, 8, 9, 10\} \setminus \{7, 8, 9\} =$?

Answer: $\underline{\qquad \{5, 6, 10\} \qquad}$

Symmetric Difference

The **symmetric difference** of sets A and B, denoted $A \oplus B$ is the set of elements that belong to A or B but not both.
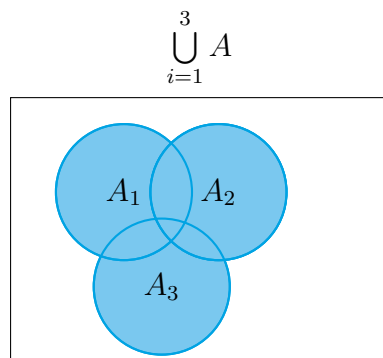
$$A \oplus B$$

$$A \oplus B = (A \cup B) \setminus (A \cap B) \qquad (4.7)$$

Example:

Union of many sets

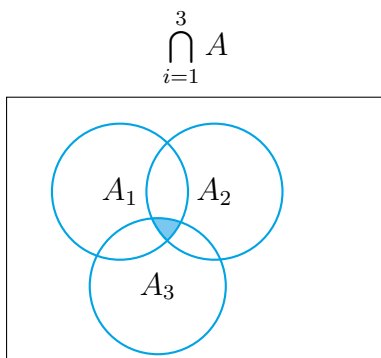We use the following notation to denote the union of sets $A_1, A_2, ..., A_n$:

$$\bigcup_{i=1}^{3} A$$

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i \qquad (4.8)$$

Example: $A_1 = \{1, 2, 3, 4\}; A_2 = \{7, 8, 9\}, A_3 = \{4, 5, 6, 7\} = ?$

Answer:
$\underline{\{1, 2, 3, 4, 5, 6, 7, 8, 9\}}$

Intersections of many sets

We use the following notation to denote the intersection of sets $A_1, A_2, ..., A_n$:

$$\bigcap_{i=1}^{3} A$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i \qquad (4.9)$$

Membership table

Just like we use truth tables for determining equality of logic statements, we can use ***membership tables*** to prove equality of sets.

1 is used to indicate that an element belongs to a set; 0 is used to indicate that an element is not in that set.

				Membership for Distributive Property			
A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	**1**	**1**	1	1	**1**
1	1	0	**1**	**1**	1	0	**1**
1	0	1	**1**	**1**	0	1	**1**
1	0	0	**0**	**0**	0	0	**0**

DeMorgan's Law

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (4.10)$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (4.11)$$

Set Problems

Example: Let A be the set of students who live within one mile of school, and B the set of students who walks to campus from home. Describe each of these sets of students:

1. $A \cap B$
2. $A \cup B$
3. $A - B$
4. $B - A$

Answer:

1. The students who live within a mile of school and walk to campus (but not take the bus).
2. The students who live within a mile of school (and walk or drive), and the students who walk to school (no matter how far it is).

3. The students who live within a mile of school but DON'T walk to school.
4. The students who walk to school but live further than a mile away.

Set Identities

Set Identities	
Identity	**Name**
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws

## 4.5 Worked Problems: Sets

1. Let A, B and C be sets. Show that $(A - B) - C$ is not necessarily equal to $A - (B - C)$.

2. Let **E** denote the set of even integers and **O** the set of odd integers. $\mathbb{Z}$ is the set of all integers. Determine these sets:

   (a) **E** $\cup$ **O**

   (b) **E** $\cap$ **O**

   (c) $\mathbb{Z}$ − **E**

   (d) $\mathbb{Z}$ − **O**

3. Show that if A and B are sets, then $A - (A - B) = A \cap B$.

4. Show that if A is a subset of B, then the powerset of A is a subset of the power set of B.

5. Show that symmetric difference follows the associative property using the following Venn diagrams. That is, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$
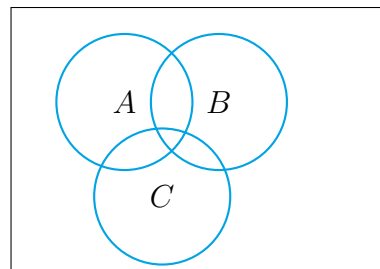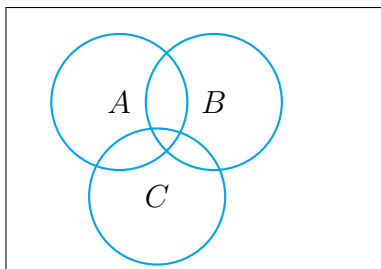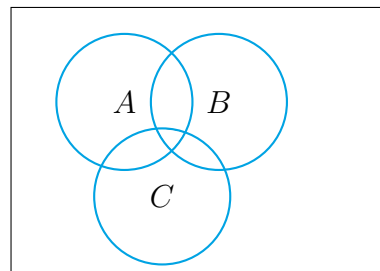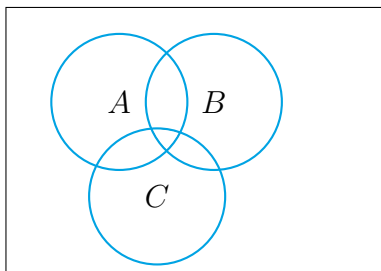
4.6   Matrices

Matrices are used throughout discrete mathematics to express relationships between elements in sets. In subsequent chapters we will use matrices in a wide variety of models. For instance, matrices will be used in models of communications networks and transportation systems. Many algorithms will be developed that use these matrix models. This section reviews matrix arithmetic that will be used in these algorithms.

Example: Social network

Here's something you might be familiar with: We start with a shape on the screen, and it transforms over time. First it moves from one place to another, then it gets

---

Definition 1:   Matrix

A matrix is a rectangular array of numbers. A matrix with m rows and n columns is called an $m \times n$ matrix. The plural of matrix is matrices. A matrix with the same number of rows as columns is called square. Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \tag{4.12}$$

$$m = 3 \tag{4.13}$$

$$n = 2 \tag{4.14}$$

$$\begin{bmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{bmatrix} \tag{4.15}$$

$$\tag{4.16}$$

What can we do with a matrix? Regardless of what is represented by the matrix, for different reasons (and different applications) we need to manipulate matrices in different ways. We can:

- Add matrices (of the same shape) (Matrix Addition)
- (Scalar Multiplication)
- Multiply matrices
- Transpose
- Determinant

- Inverse

Properties matrices can have:

- Square
- Symmetric
- Column/Row (Vector)

4.6.0.1 Adding Matrices

$$
\begin{bmatrix} x_{0,0} & x_{0,1} & x_{0,2} \\ x_{1,0} & x_{1,1} & x_{1,2} \\ x_{2,0} & x_{2,1} & x_{2,2} \end{bmatrix} + \begin{bmatrix} y_{0,0} & y_{0,1} & y_{0,2} \\ y_{1,0} & y_{1,1} & y_{1,2} \\ y_{2,0} & y_{2,1} & y_{2,2} \end{bmatrix} = \begin{bmatrix} x_{0,0} + y_{0,0} & x_{0,1} + y_{0,1} & x_{0,2} + y_{0,2} \\ x_{1,0} + y_{1,0} & x_{1,1} + y_{1,1} & x_{1,2} + y_{1,2} \\ x_{2,0} + y_{2,0} & x_{2,1} + y_{2,1} & x_{2,2} + y_{2,2} \end{bmatrix} \tag{4.17}
$$

Example: Adding 2 matrices

$$
\begin{bmatrix} 5 & 4 & 3 \\ 3 & 3 & 3 \\ -1 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 15 & 30 \\ 5 & 20 & 35 \\ 10 & 25 & 35 \end{bmatrix} = \begin{bmatrix} 5 & 19 & 33 \\ x_{1,0} + y_{1,0} & x_{1,1} + y_{1,1} & x_{1,2} + y_{1,2} \\ x_{2,0} + y_{2,0} & x_{2,1} + y_{2,1} & x_{2,2} + y_{2,2} \end{bmatrix} \tag{4.18}
$$

Example: Adding two different matrices

$$
\begin{bmatrix} 5 & 4 & 3 \\ 3 & 3 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 15 \\ 5 & 20 \\ 10 & 25 \end{bmatrix} = ??? \tag{4.19}
$$

4.6.0.2 Multiplying Matrices

Definition 2: Matrix multiplication

$$
c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}
$$

Number of columns in A needs to equal number of rows in B.

$$
\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix} \times \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{bmatrix} = \tag{4.20}
$$

$$
= \begin{bmatrix} a_1a_2 + b_1d_2 + c_1g_2 & a_1b_2 + b_1e_2 + c_1h_2 & a_1c_2 + b_1f_2 + c_1i_2 \\ d_1a_2 + e_1b_2 + f_1h_2 & d_1b_2 + e_1e_2 + f_1h_2 & d_1c_2 + e_1f_2 + f_1i_2 \end{bmatrix} \tag{4.21}
$$

$$\begin{bmatrix} 5 & 4 & 3 \\ 3 & 3 & 3 \end{bmatrix} \times \begin{bmatrix} 0 & 15 \\ 5 & 20 \\ 10 & 25 \end{bmatrix} = \begin{bmatrix} 20 + 30 & 75 + 80 + 5 \\ 15 + 30 & 45 + 60 + 5 \end{bmatrix} \tag{4.22}$$

(We know the final matrix will be 2 x 2, because A has 2 rows, B has 2 columns, and the final matrix of a matrix multiplication has the same number of rows as A, and the same number of cols as B)

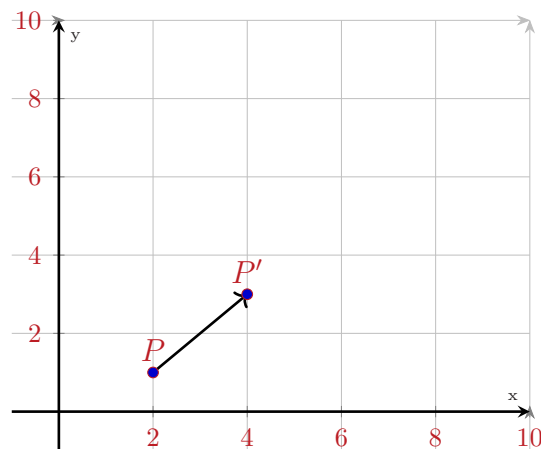#### 4.6.0.3   Zero-One Matrices

Example: Sending email

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \tag{4.23}$$

Boolean operations on zero-one matrices: And,OR, XOR, ...

4.6.1   Situating Example: Graphics

Problem: I have this picture, about as simple as it can be:

I have a point on a screen, and I want to move it to another place. Right now, I'm just working with a point, but you could imagine that it's a shape of some kind (a sprite!).

How do computers deal with this?

The point is represented by a matrix! To represent a change in the location of the point, we apply what's called a ***transformation***, or specifically in this case, a ***translation***.

If the point is originally located at (x, y), the new position is (x', y'), where:

$$x' = x + t_x \tag{4.24}$$
$$y' = y + t_y \tag{4.25}$$

That is: the new x is the original x plus the change in the x-direction, and the new y is the original y plus the change in the y-direction.

Here's where matrices come in: we model the point as a ***column vector***. The change is what we call the ***translation vector***, also modeled as a column vector. Therefore we get something like this:

$$P = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, P' = \begin{bmatrix} x'_1 \\ y'_y \end{bmatrix}, T = \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{4.26}$$

Per our earlier statement that $p' = p + t$:

$$P' = P + T \tag{4.27}$$
$$\begin{bmatrix} x'_1 \\ y'_y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \tag{4.28}$$
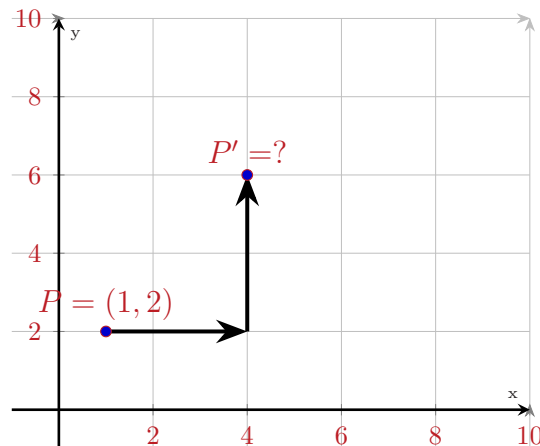$$= \begin{bmatrix} x_1 + t_x \\ y_1 + t_y \end{bmatrix} \tag{4.29}$$
$$\implies x'_1 = x_1 + t_x \tag{4.30}$$
$$y'_1 = y_1 + t_y \tag{4.31}$$

Putting numbers to this:

I have a point at $(1, 2)$. I need to move it 3cm horizontally, and 4cm vertically. What's the final position?
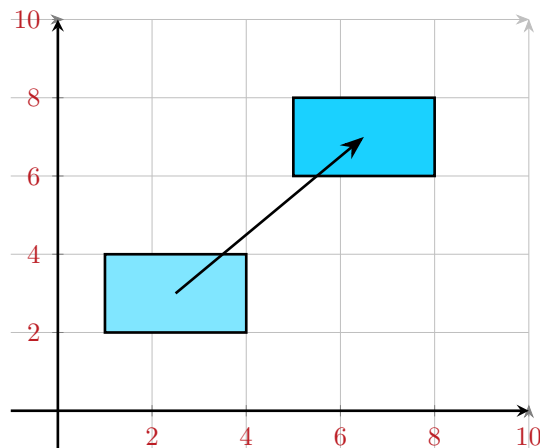
$$P' = P + T \tag{4.32}$$

$$\begin{bmatrix} x'_1 \\ y'_y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{4.33}$$

$$\begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 + t_x \\ y_1 + t_y \end{bmatrix} \tag{4.34}$$

$$\tag{4.35}$$

The new location of the point is $4, 6$.

Let's make translation a little more complicated:

What's trickier here is that I have more points. To move the entire rectangle, I need to do what's called a ***rigid body transformation***. That's a fancy way of saying that we want the rectangle to keep it's shape when we move it: it's a *rigid body*. Now, we need to somehow represent this rigid body, and I want to do it as a series of points:

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 4 & 4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} t_x & t_y \end{bmatrix} \tag{4.36}$$

$$\tag{4.37}$$

What's t_x? ___**4**___ t_y ___**4**___ ?

Wait: can we do this? The

- First, I represented each point in a row, instead of a column. That gives me two columns, an x- and y- column. It doesn't matter what I do here: I could represent x's as a row, and y's as another row, so we have a $2 \times n$ matrix, where n is the number

points we have. Or, I could just represent each point as a 1×2 matrix, and apply the same transform to each point.
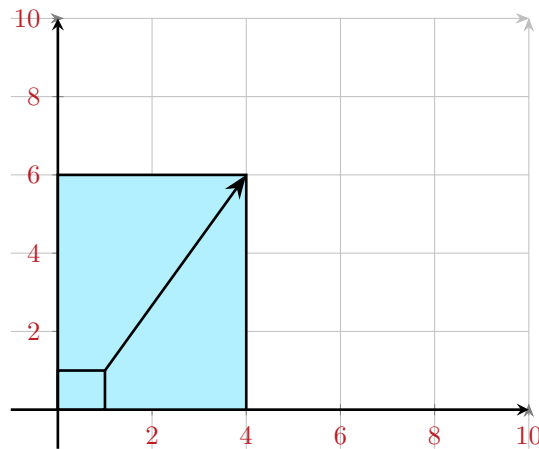
- Now, I can make T as arbitrarily long as I need it by just duplicating rows of T.

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 4 & 4 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} t_x & t_y \\ t_x & t_y \\ t_x & t_y \\ t_x & t_y \end{bmatrix} \tag{4.38}$$

$$\tag{4.39}$$

Translation is pretty straightforward. Let's do something a little more complicated.

Scaling

$$x' = x \cdot s_x, y' = y \cdot s_y \tag{4.40}$$

This means that to scale a polygon, we'll multiple each vertex by the ***scaling factors*** s_x and s_y. s_x is the amount to scale in the horizontal direction, and s_y is the amount to scale in the vertical direction.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \tag{4.41}$$

which, similar to our translation transfromation, can be summarized as:

$$P' = S \cdot P \tag{4.42}$$

Where does those 0s in the scaling matrix above come from? Let's do the math:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (s_x \cdot x + 0 \cdot y) \\ (0 \cdot x + x_y \cdot y) \end{bmatrix} \tag{4.43}$$

In the above image, we have a polygon with the following coordinates:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{4.44}$$

We want to scale it by 4 in the x direction, and 6 in the y direction: $s_x = 4, s_y = 6$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} (0 \cdot 4 + 0 \cdot 0) & (0 \cdot 0 + 0 \cdot 6) \\ (0 \cdot 4 + 1 \cdot 6) & (0 \cdot 0 + 0 \cdot 6) \\ (1 \cdot 4 + 1 \cdot 0) & (1 \cdot 0 + 1 \cdot 6) \\ (1 \cdot 4 + 0 \cdot 0) & (1 \cdot 0 + 0 \cdot 6) \end{bmatrix} \tag{4.45}$$

$$= \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 4 & 6 \\ 4 & 0 \end{bmatrix} \tag{4.46}$$

If we reconcile these final coordinates with the original figure, you see they correspond.

Composite Transforms: Scale & Translation
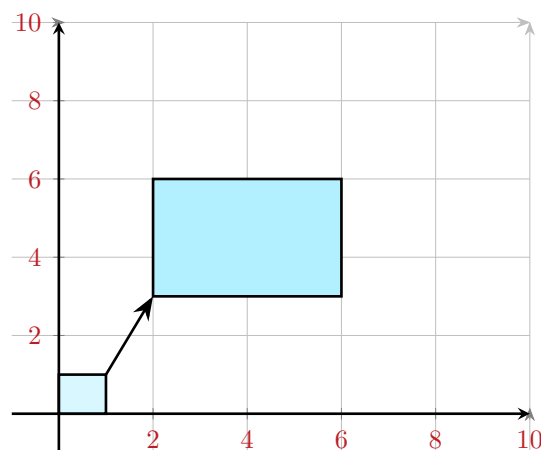
If we want to scale and translate, you can take the points, and first apply one transform, and then the other. But, that's not terribly efficient. We can do multiple transforms at once, combining a scale and a translate:

$$\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \tag{4.47}$$

Okay, so what do we know? I claim that the above matrix applied to a rigid body will reflect it around the x-axis. To do this, we multiply each point by the transform matrix. But, this matrix is 3x3, and our points are 1x2. How do we deal with that?

Answer: **Pad with 1s or 0s**

Instead of representing a point by a 2x1 matrix, we represent it by a 3x1 matrix:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The above figure depicts a translation of (2, 3), and a scaling of (3, 4).

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Or, you can write it as $P' = S(s_x, s_y) \cdot T(t_x, t_y) \cdot P$.

Let's do the example for the point (1, 1) for the above scaling/translating:

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 4x + 2 \\ 3y + 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 6 \\ 1 \end{bmatrix} \text{ (Substituting } x = 1, y = 1)$$

(We know the final matrix will be 3 x 1, because A has 3 rows, B has 1 column, and the final matrix of a matrix multiplication has the same number of rows as A, and the same number of cols as B)

We can draw the conclusion that for each point in our shape, we have a formula for the new location:

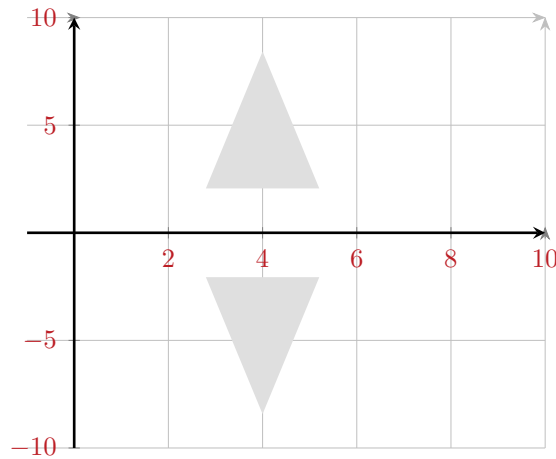$$x' = xs_x + t_x$$
$$y' = xs_y + t_y$$

Prove it by transforming the other points in the diagram above.

Matrix multiplication is associative, but not necessarily communtative:

$$A \cdot B \cdot C = (A \cdot B) \cdot C = A \cdot (B \cdot C)$$
$$A \cdot B \neq B \cdot A$$

Reflection

To **reflect** an object, we transform to produce a mirror image. The picture above shows a reflection around the x-axis, which is the axis of reflection.

The transform for reflecting around the x-axis is below:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{4.48}$$

To flip around the y axis:

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.49}$$

4.7   Summary

Today, we talked about sets and matrices. When it comes to sets, the biggest thing is figuring out the size of a set. We use sets of known sizes to help us determine the sizes of other sets:

Is that other set bigger than, smaller than, or the same size as this set? Now that we've talked about sets, you can likely see their direct application to logic and functions. In fact, set theory arose out of trying to reason about reasoning, that is what logic was.

We also talked about matrices. I used computer graphics to motivate our matrix manipulation exercises. In addition to learning about matrices, you learned a little bit about graphics as well!

Readings for NEXT week:

Rosen, Chapter 4.1, 4.2, 4.3, 4.4

Divisibility and Modular Arithmetic, Integer Representations and Algorithms,

Primes and Greatest Common Divisors, Solving Congruences Solving Congruences

---