CS 5002: Discrete Structures **Fall 2018**

Lecture 3: September 20, 2018 [1]

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Readings for this week:
Rosen, Chapter 2.4: Sequences and Summations

3.1 Overview

1. Sequences
2. Summations
3. Recurrences
4. Closed forms
5. Applications
6. Problem Solving/Examples

3.2 Introduction

Today, we're going to talk about some topics that demonstrate very strongly the practical nature of discrete math. The topics we talk about today reflect patterns we find in nature, and applications to our own personal lives.

We'll start with some basics and definitions. But then we'll move into some problem solving: we're going to solve some basic summations and such, but we'll also solve some more complex problems.

3.3 Review

- inverse functions
- compositions
- floor and ceiling

3.4 Situating Problem Introduction

Let's work through a problem y'all are likely to face soon.

You get a job offer: Yay! It seems pretty good, but you want to analyze it a bit.

Here are the details:

- Your base salary is $\$100,000$
- You get a raise of 5% every year.

This sounds great! So, you want to know: how much will I have earned after 5 years at this company? Maybe you want to figure out if you can pay off your student loans or buy a house or something.

How do we calculate this? Well, we know math. We can do this!

	Salary	5% of salary
Year 1:	**$100,000**	**5,000**
Year 2:	**$105,000**	**5,250**
Year 3:	**$110,250**	**5,512.50**
Year 4:	**$115,762.50**	**5,788.13**
Year 5:	**$121,550.63**	**?**
Total:	**$552,563.13**	

Okay, this is great. We know the answer! But let's say that I'm really looking a few more years out. This table works, but it's kinda clunky.

I'll make an observation: This table is really equivalent to an equation:

I can re-write this as:

$$100000 + [100000 + 5000] + [105000 + \ldots] \ldots \tag{3.1}$$

And, because this is a math class, I'm getting annoyed at writing all these numbers. Let me replace these numbers with variables and rewrite it as such:

$$a_0 + (a_0 + (0.05 \cdot a_0)) + (a_1 + (0.05 \cdot a_1)) + (a_2 + (0.05 \cdot a_2)) + \ldots \tag{3.2}$$

It's getting a little less cumbersome, but I'll simplify it one more time by introducing the summation notation:

Summation Notation Example

$$\begin{aligned} \sum_{j=0}^{5} 2j &= (2 \cdot 0) + (2 \cdot 1) + (2 \cdot 3) + (2 \cdot 4) + (2 \cdot 5) && (3.3) \\ &= 0 + 2 + 4 + 6 + 8 + 10 && (3.4) \\ &= 30 && (3.5) \\ & && (3.6) \end{aligned}$$

Applied to our formula:

$$a_0 + \sum_{j=1}^{4} a_{j-1} + (a_{j-1} \cdot 0.05) = a_0 + \sum_{j=1}^{4} (a_{j-1} \cdot 1.05) \tag{3.7}$$

Which, if we were to expand out, would be the same as:

$$a_0 + ((a_1 \cdot 1.05) + (a_2 \cdot 1.05) \ldots (a_4 \cdot 1.05)) \tag{3.8}$$

Note: a_0 is pulled out because it doesn't match our pattern inside the summation. The first year, we just earn a_0; the rest of the years, salary increases by a percentage.

What is a_0? a_1?

<u>$a_0 = 100000$</u> <u>$a_1 = 105000$</u>

We know a_0: it's our original salary! If we plug a_0 in the first term, we can calculate all the other terms.

Therefore, to answer our original question: How much will we have made over those 5 years?

<u>**552,563.13**</u>

3.5 Sequences

Definition 1: Sequence

Collection of ordered elements.
We use a_n to denote the n^{th} term of the sequence.

Examples of sequences:

$$\begin{gathered}\{1, 3, 5, 7, 9, \ldots\} \\ \{2, 4, 6\}\end{gathered} \tag{3.9}$$

3.5.1 Geometric Sequence

$$a, ar, ar^2, ar^3, ar^4, \ldots ar^n \tag{3.10}$$

Example:

$$\begin{aligned}\{b_n\} : b_n &= (-1)^n \\ b_0 &= (-1)^0 \\ b_1 &= (-1)^1 \\ b_2 &= (-1)^2 \\ \{b_0, b_1, b_2&, \ldots\} \\ \{1, -1, 1, &-1, 1, \ldots\}\end{aligned} \tag{3.11}$$

This is equivalent to $f(x) = ar^x$, which is an **exponential** function.

3.5.2 Arithmetic Sequence

$$a, a + d, a + 2d, a + 3d, \ldots a + nd \tag{3.12}$$

This is equivalent to $f(x) = dx + a$, which is called a **linear function** .

(Why? Look at a graph of the function).

$$
\begin{aligned}
\{s_n\} : s_n &= -1 + 4n \\
s_0 &= \underline{\quad -1 \quad} \\
s_1 &= \underline{\quad 3 \quad} \\
s_2 &= \underline{\quad 7 \quad} \\
s_3 &= \underline{\quad 11 \quad} \\
&\vdots \\
\{s_0, s_1, \ldots s_n\} \\
\{&\underline{\qquad\qquad -1, 3, 7, 11, \ldots \qquad\qquad}\}
\end{aligned} \tag{3.13}
$$

An alternative way to think about this:

$$
\begin{aligned}
f(x) &= \underline{\quad 4x - 1 \quad} \\
f(1) &= \underline{\quad 3 \quad} \\
f(2.5) &= \underline{\quad 9 \quad} \\
&\vdots
\end{aligned} \tag{3.14}
$$

Back to our problem of interest:

$$\sum_{j=1}^{5} (a_{j-1} \cdot 1.05) \tag{3.15}$$

Is it geometric or arithmetic? **geometric**

Is it discrete or continuous? **discrete**

Strings

Sequences are sometimes called **strings**.

Empty string: λ

Length of a string: Number of terms or elements in the string.

Length of λ: 0

Side note that's interesting: time is continuous, but anything that we measure over time is discrete. Why? (Because we have to sample– take a reading every so often!)

Definition 2: Recurrence Relation

Given a sequence $\{a_n\}$, a_n is expressed in terms of 1 or more previous terms in the sequence.

3.5.3 Recurrence Relations

A sequence is a ______**solution**______ of a recurrence relation if the ______**terms**______ satisfy the recurrence relation.

Example:

$$\begin{aligned}\{a_n\} &: a_n = a_{n-1} + 3 \\ a_0 = 2 &: \underline{\ \mathbf{\{5,\ 8,\ 11,\ \ldots\}}\ } \\ a_0 = 6 &: \underline{\ \mathbf{\{9,\ 12,\ 15,\ \ldots\}}\ }\end{aligned} \tag{3.16}$$

We've shown two solutions to the recurrence relation.

Solutions are unique depending on the __**initial condition**__ .

3.5.4 Closed Formula

We now know that given an initial condition, we can find a sequence that is a solution to a recurrence. But, sometimes it's awkward to write out a sequence, and using formulas are just more convenient. To ***solve*** the recurrence, we have to find a formula for the solution. So, in addition to finding a solution, we look for a __**closed formula**__ .

That is, instead of finding the list of terms (the sequence) that we add together, can we find a formula that calculates the final result?

Example:

Given this ______**recurrence**______ :

$$a_n = 2a_{n-1} - a_{n-2}, \text{ for } n = 2, 3, 4, \ldots \tag{3.17}$$

Is this a solution to the above recurrence?

$$\{a_n\} : a_n = 3n \forall \text{ non-negative integers } n \tag{3.18}$$

Yes, it's a solution. Below are the first few elements of the sequence:

$$\{6, 9, 12, 15, \ldots\}, \text{ for } n = 2, 3, 4, \ldots \tag{3.19}$$

And,

$$3n = 2(3(n-1)) - 3(n-2), \text{ for } n = 2, 3, 4, \ldots \tag{3.20}$$

$$2(3(n-1)) - 3(n-2) = 3n = a_n, \text{ for } n = 2, 3, 4, \ldots \tag{3.21}$$

But, it's a little awkward to write.

Now, I propose this is a closed formula for the above recurrence, given initial conditions of $a_2 = 3; a_3 = 6$:

$$a_n = 6n - 3(n-1) \tag{3.22}$$

We can check this by plugging in numbers. Since this is defined for $n > 1$, let's plug in some numbers:

$$\begin{aligned} n = 4 : a_4 &= 6 \cdot 4 - 3 \cdot 3 \rightarrow 15 \text{ checks out with our sequence above in 3.19} \\ n = 6 : a_6 &= 6 \cdot 6 - 3 \cdot 5 \rightarrow 21 \text{ check my math!} \end{aligned} \tag{3.23}$$

We know how to check if something is a closed formula for a recurrence— it's fairly straightforward. But, how do we come up with a ______**solution**______ for a recurrence? To see the derivation of the closed formula in 3.22, see 3.6.3.

For now, ______**Iteration!**______ : We'll learn more techniques later this semester.

For the example above:

Let's start with the ______**first term**______ and ______**initial condition**______ . And then we work through, until we get to a a_n where we can deduce a closed formula.

Recurrence: $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, 4, \ldots$

Initial condition: $a_1 = 2$

$$\begin{aligned} a_2 &= 2 + 3 \\ a_3 &= (2 + 3) + 3 = 2 + 3 \cdot 2 \\ a_4 &= (2 + 2 \cdot 3) = 2 + 3 \cdot 3 \\ &\vdots \\ a_n &= a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1) \end{aligned} \tag{3.24}$$

Or you can do it backwards!

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 \to a_{n-2} + (2 \cdot 3) \\
&= ((a_{n-3} + 3) + 3) + 3 \to a_{n-3} + (3 \cdot 3) \\
&\vdots \\
a_n &= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1)
\end{aligned}
\tag{3.25}
$$

Define: iteration, forward substitution, backward substitution.

What have we been doing with our salary example so far? **(a little of both)**

Let's come up with a closed formula for the calculation we've done by hand so far:

$$a_0 + \sum_{j=1}^{n} (a_{j-1} \cdot 1.05)$$

Let's work out the first few terms; replacing rate with a variable

$$
\begin{aligned}
a_1 &= ra_0 \\
a_2 &= ra_1 = ra_0 = r^2 a_0 \\
a_3 &= ra_2 = ra_1 = r^3 a_0 \\
&\vdots \\
a_n &= r^n a_0 \\
&\text{Replacing } r \text{ and } a_0 \\
a_n &= 1.05^n \cdot 100000 \\
a_4 &= ??
\end{aligned}
\tag{3.26}
$$

This is a closed formula for the recurrence part; but it doesn't handle the summation.

$$
\begin{aligned}
&a_0 + \sum_{j=1}^{n} (a_{j-1} \cdot 1.05) \\
\Rightarrow\ &a_0 + \sum_{j=1}^{n} r^j a_0
\end{aligned}
\tag{3.27}
$$

We can also come up with a closed form for the summation, but we'll see this later.

3.5.5 An Aside: Recurrences in programming

Recurrences are important in computer science. This idea comes up in what we call **recursive** functions.

I can write a Python function to calculate the amount I've earned after n years:

```python
def calcAmount(base, raise, numYears):
  if (numYears == 1):
    return base;
  prevYr = calcAmount(base, raise, numYears-1)
  return (prevYr + (prevYr * 1.05))
```

3.5.6 Special Sequences

3.5.6.1 Fibonacci

The Fibonacci[2] sequence is a special sequence.

It starts with the following initial condition: $f_0 = 0; f_1 = 1$. Then, a Fibonacci number f_n is defined:

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2 \tag{3.28}$$

3.5.6.2 Prime Numbers

Sequences that Cannot be Represented Easily

It needs to be mentioned that there are many sequences that are not easily represented either as a closed form or as a recurrence.

The sequence of prime numbers is one such sequence.

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 27, 31, \ldots \tag{3.29}$$

We do not have a "easy" closed form for the n^{th} prime number (there are some really complicated ones that are not worth going into). Same holds for recurrences: the known ones are quite complicated.

[2]Fibonacci: http://memolition.com/2014/07/17/examples-of-the-golden-ratio-you-can-find-in-nature/

3.5.6.3 Sequences of Letters and Grammars

Let us assume that there are two letters A and B. We can define sequences of "words" over these letters:

$$B, BAB, BABAB, BABABAB, \ldots \tag{3.30}$$

The recurrence for such sequences is often called a "grammar" and written slightly differently (after the work of computer scientists and linguists such as Chomsky, Naur, ...).

The recurrence is given by

$$a_n = a_{n-1}AB \tag{3.31}$$

which means, that the n^{th} element of the sequence is the $n-1^{th}$ element with the characters AB tacked on to the end. The base case is given by

$$a_1 = B \tag{3.32}$$

Here is a more complex (context-free) recurrence relation:

$$b_n = Bb_{n-1}B \tag{3.33}$$

with $b_1 = A$ as the base case.

What is the sequence obtained here?

Answer: $A, BAB, BBABB, BBBABBB, \ldots$.

Interestingly, the theory behind formal grammars is the basis for how we write compilers for programming languages. It is one of the most important applications of theoretical computer science or discrete mathematics to computer science.

3.6 Summations

Okay, I've already introduced the idea of a summation, or using this funky symbol to represent combining a bunch of terms of a sequence together. Let's get a little more into summations.

Example:

Definition 3: Summation

$$\sum_{j=x}^{n} \Rightarrow a_x + a_{x+1} + a_{x+2} \ldots + a_n \tag{3.34}$$

$$\begin{aligned} \sum_{k=3}^{8} 2k &= \underline{(2\cdot 3)} + \underline{(2\cdot 4)} + \underline{(2\cdot 5)} + \underline{(2\cdot 6)} + \underline{(2\cdot 7)} + \underline{(2\cdot 8)} \\ &= \underline{\mathbf{6 + 8 + 10 + 12 + 14 + 16}} \\ &= \underline{\mathbf{66}} \end{aligned} \tag{3.35}$$

Might also see $\sum_{i=1}^{10} a_i$, particularly when the sum is written inline/ in text.

It's helpful to have some ways to manipulate these summations.

Let's say we have the following two summations.

$$\begin{gathered} \sum_{j=1}^{5} j \\ \sum_{k=0}^{4} (k+1) \end{gathered} \tag{3.36}$$

A summation can also be defined recursively, which can be helpful. Here, we're pulling out the last term of the summation:

$$\sum_{j=a}^{n} f(j) = f(n) + \sum_{j=a}^{n-1} f(j) \tag{3.37}$$

We can also pull out the first term of the summation:

$$\sum_{j=a}^{n} f(j) = f(a) + \sum_{j=a+1}^{n} f(j) \tag{3.38}$$

See 3.6.2 for an example.

Replacing summations with a closed form.

We talked about closed formulas earlier; we've got a similar thing for summations called <u>**closed forms**</u> . In the <u>**closed form**</u> , the expression does not use any subscripted summations or elements. In fact, there are some helpful formulas that come into use:

See the rest in your book, but here are some examples:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{3.39}$$
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.40}$$
$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \tag{3.41}$$

(Also, Wikipedia is a great reference: https://en.wikipedia.org/wiki/Summation)

We talked about geometric sequences earlier; they arise a lot. In fact we see it in the example we've been using:

$$a_0 + \sum_{j=1}^{n}(a_{j-1} \cdot 1.05) = a_0 + \sum_{j=1}^{n} r^j a_0$$
$$a_0 + \sum_{j=1}^{n} r^j a_0 = a_0 + \sum_{j=0}^{n-1} r^{j+1} a_0 \text{ (changing limits to fit the closed form 3.41)}$$

This also lets us move the a_0 term into the summation, and move the upper limit to n: (see 3.38 and 3.6.2)

$$\begin{aligned} a_0 \sum_{j=0}^{n} r^j &= a_0 \frac{1-r^{n+1}}{1-r} \text{ (using closed form from 3.41)} \\ &= a_0 \frac{1-(1.05)^{n+1}}{1-1.05} \\ &= 100000 \cdot \frac{1-(1.05)^{n+1}}{1-1.05} \end{aligned} \tag{3.42}$$

Evaluate this at $n = 5$ should give us the amount of money we will have made after 5 years:

$$\frac{1-(1.05)^{n+1}}{1-1.05} \cdot 100000 = 552,563.13 \tag{3.43}$$

Example: Double Sums

In computer science, we frequently come across double sums, or a summation of a summation. It looks a little scary, and can be a little hairy, but if we keep our wits about ourselves, we can untangle them.

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=2}^{4} i+j &= \sum_{i=1}^{3}(i+2)+(i+3)+(i+4) \\
&= \sum_{i=1}^{3} 3i+9 \\
&= (3+9)+(3\cdot 2+9)+(3\cdot 3+9) \\
&= 12+15+18 \\
&= 45
\end{aligned} \tag{3.44}$$

$$\begin{aligned}
\sum_{1\le j,k\le 3} a_j b_k = a_1b_1 + a_1b_2 + a_1b_3 \\
+ a_2b_1 + a_2b_2 + a_2b_3 \\
+ a_3b_1 + a_3b_2 + a_3b_3
\end{aligned} \tag{3.45}$$

Here, the $\sum$ symbol denotes a sum over all combinations of i, j, the relevant indices.

You can see it's the same as a sum of sums, where each summation gets a different index.

$$\begin{aligned}
\sum_{1\le j,k\le 3} a_j b_k &= \sum_{1\le j\le 3}\sum_{1\le k\le 3} a_j b_k \\
&= \sum_{1\le j,k\le 3} a_jb_1 + a_jb_2 + a_jb_3 \\
&= a_1b_1 + a_1b_2 + a_1b_3 \\
&+ a_2b_1 + a_2b_2 + a_2b_3 \\
&+ a_3b_1 + a_3b_2 + a_3b_3
\end{aligned} \tag{3.46}$$

3.6.1 Products

Just like we can sum over a number of terms, we can take the product of a number of terms.

Example:

$$\prod_{i=m}^{n} a_i \tag{3.47}$$

This represents the product $a_m \cdot a_{m+1} \cdot \ldots a_{n-1} \cdot a_n$.

Example:

$$\prod_{i=1}^{5} i = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \\ = 120 \tag{3.48}$$

This is a special product called a ____**factorial**____ , defined such that $f(n) = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$.

Factorials come up quite a bit, so it's helpful to recognize them and be familiar manipulating them:

$$\frac{(n+3)!}{n!} = \frac{(n+3)(n+2)(n+1)(n!)}{n!} \tag{3.49}$$

3.6.2 Review our original problem

- Wanted to sum up how much we'd earn over time.
- The amount was a summation of a recurrence.
- We found a closed form for the recurrence.
- Then found a closed form for the summation.
- Then, at the end, had a formula that provided us with the amount earned over n years, by just plugging the rate change, the initial value, and n.

In the following few lines, I've combined all the steps we took throughout the discussion above.

We started with a summation:

$$a_0 + \sum_{j=1}^{4} (a_{j-1} \cdot 1.05) \tag{3.50}$$

$$\tag{3.51}$$

We then used ***forward iteration*** to find the ***closed formula*** of the ***recurrence*** inside the ***summation***.

$$\begin{aligned} &\text{Working out the first few terms; replacing rate with variable } r \\ &a_1 = ra_0 \\ &a_2 = ra_1 = ra_0 = r^2a_0 \\ &a_3 = ra_2 = ra_1 = r^3a_0 \\ &\vdots \\ &a_n = r^n a_0 \\ &\text{Replacing } r \text{ and } a_0 \\ &a_n = 1.05^n \cdot 100000 \end{aligned} \tag{3.52}$$

Using this closed formula for the recurrence, we plug it into the summation to simplify:

$$a_0 + \sum_{j=1}^{n}(a_{j-1} \cdot 1.05) \Longrightarrow a_0 + \sum_{j=1}^{n} \quad r^j a_0 \tag{3.53}$$

This closed formula for the recurrence part didn't handle the summation, so we used a known ***closed form*** for a summation, to find a closed form for our particular summation. Here, we're taking our original formula, changing the limits, then plugging a known closed form (3.41).

First: change the limits, because the closed form starts at $j = 0$:

$$a_0 + \sum_{j=1}^{n}(a_{j-1} \cdot 1.05) = a_0 + \sum_{j=1}^{n} r^j a_0 \tag{3.54}$$

$$a_0 + \sum_{j=1}^{n} r^j a_0 \Rightarrow a_0 + \sum_{j=0}^{n-1} r^{j+1} a_0 \tag{3.55}$$

$$\tag{3.56}$$

Now, what we have is equivalent to:

$$a_0 + \sum_{j=0}^{n-1} r^{j+1} a_0 = a_0 + \left(r^1 a_0 + r^2 + r^3 a_0 \ldots r^n a_0\right) \tag{3.57}$$

$$\Rightarrow r^0 a_0 + \left(r^1 a_0 + r^2 + r^3 a_0 \ldots r^n a_0\right) \tag{3.58}$$

This means, we can move that first a_0 term the we've been adding to the outside of the summation inside the summation:

$$\left(r^0 a_0 + r^1 a_0 + r^2 + r^3 a_0 \ldots r^n a_0\right) \tag{3.59}$$

When we do that, we can change r^{j+1} to r^j, and change the upper limit to n:

$$\left(r^0 a_0 + r^1 a_0 + r^2 + r^3 a_0 \ldots r^n a_0\right) = \sum_{j=0}^{n} r^j a_0 \tag{3.60}$$

If you need, take a couple minutes to convince yourself 3.57 equals 3.60.

$$\sum_{j=0}^{n} r^j a_0 = a_0 \sum_{j=0}^{n} r^j \text{ (see Summation identities)} \tag{3.61}$$

$$a_0 \sum_{j=0}^{n} r^j = a_0 \frac{1 - r^{n+1}}{1 - r} \text{ (using closed form from 3.41)} \tag{3.62}$$

$$= a_0 \frac{1 - (1.05)^{n+1}}{1 - 1.05} \tag{3.63}$$

$$= 100000 \cdot \frac{1 - (1.05)^{n+1}}{1 - 1.05} \tag{3.64}$$

$$\tag{3.65}$$

Evaluate this at $n = 5$ should give us the amount of money we will have made after 5 years:

$$\frac{1 - (1.05)^{n+1}}{1 - 1.05} \cdot 100000 = 552,563.13 \tag{3.66}$$

So: we went from a general idea of how to sum up an increasing salary over a given number of years, and derived a closed formula to calculate it given just a random n. That means, we can quickly determine that we'll have earned \$1 after 10 years of working. We can also do things like change the base salary (a_0, $100,000$ in our example) or the raise amount (r, 5% in our example) to compare different offers.

$$a_0 + \sum_{j=1}^{4} (a_{j-1} \cdot (1 + r)) \Longrightarrow a_0 \frac{1 - (1 + r)^{n+1}}{1 - (1 + r)} \tag{3.67}$$

3.6.3 Example of using forward iteration to derive closed formula for recurrence

Here, I provide the derivation of the closed formula for the recurrence in 3.22.

We started with the recurrence:

$$a_n = 2a_{n-1} - a_{n-2}, \text{ for } n = 2, 3, 4, \ldots \tag{3.68}$$

We validated that this sequence is a solution of the recurrence:

$$\{a_n\} : a_n = 3n \forall \text{ non-negative integers } n \tag{3.69}$$

And I stated this was the closed form:

$$a_n = 6n - 3(n - 1) \tag{3.70}$$

I derived this by using forward iteration. Remember: our goal is to find a formula that expresses a_n in terms of ONLY our initial conditions a_0 and a_1.

$$
\begin{aligned}
a_0 &= ? \\
a_1 &= ? \\
a_2 &= 2a_1 - a_0 \\
a_3 &= 2a_2 - a_1 = 2(2a_1 - a_0) - a_1 \\
&= 4a_1 - 2a_0 - a_1 \\
&= 3a_1 - 2a_0 \\
a_4 &= 2a_3 - a_2 = 2(3a_1 - 2a_0) - (2a_1 - a_0) \\
&= 6a_1 - 4a_0 - 2a_1 + a_0 \\
&= 4a_1 - 3a_0 \\
a_5 &= 2a_4 - a_3 = 2(4a_1 - 3a_0) - (3a_1 - 2a_0) \\
&= 8a_1 - 6a_0 - 3a_1 + 2a_0 \\
&= 5a_1 - 4a_0 \\
a_6 &= 2a_5 - a_4 = 2(5a_1 - 4a_0) - (4a_1 - 3a_0) \\
&= 10a_1 - 8a_0 - 4a_1 + 3a_0 \\
&= 6a_1 - 5a_0 \\
&\vdots \\
a_n &= na_1 - (n-1)a_0
\end{aligned}
$$

At each step, I wrote out the value of the term for that element. Then, I replaced the values I derived for earlier terms in that statement. Then, I simplified the statement. Once I get to about a_5 or a_6, I could see a pattern. I then wrote out the final statement for a_n in general, then checked that against previous elements.

Now, once I define my initial condition, I have a unique solution for the recurrence. When $a_0 = 3$ and $a_1 = 6$, the final closed formula is $a_n = 6n - 3(n-1)$.

3.7 Example Problems

(more coming)

3.8 Applications of Sequences, Summations and Recurrences

Random Number Generator

Here is an interesting fact: sequences defined using recurrence relations are used inside computers to generate pseudo random numbers. The most common sequence used is a *Linear Congruential Generator.*

The eighth International Conference on Sequences and Their Applications (SETA'14)

Readings for NEXT week:

Rosen, Chapter 2.1, 2.2, 2.5, 2.6
Sets, Set Operations, Cardinality of Sets, Matrices