A8: COMBINATORICS

I'm not counting any chickens. —Jeff Bridges

Course: CS 5002
Fall 2018
Due: 11 Nov 2018, Midnight

OBJECTIVES

After you complete this assignment, you will be comfortable with:

- Calculating binomials
- Counting combinations and permutations
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RELEVANT READING

Rosen:

- 6.1: Basics of Counting
- 6.2: Pigeonhole Principle
- 6.3: Permutations and Combinations
- 6.4: Binomial Coefficients and Identities
- 6.5: Generalized Permutations and Combinations

NEXT WEEK'S READING

- Algorithms Unlocked (Cormen), Chapters 1 and 2
- Rosen, Chapter 3.2 (Growth of Functions)

Optional:

- Rosen, Chapter 1
- Rosen, Chapter 3.3 (Complexity of Functions)
- CLRS Chapter 3 (Growth of Functions)
- CLRS Chapter 1 (The role of algorithms in computing)

EXERCISES

Question 1

(a)

$$\binom{16}{8}$$

(2)

(b)

$$\binom{42}{40}$$

(2)

Points: _____ out of 4

(c)

$$\binom{12}{7}$$

(2)

(d)

$$\binom{5}{4}$$

(2)

Question 2

For the following problems, use the Binomial Theorem to expand and simplify.

(a) $(a+b)^6$ (3)

(b) $(a+b)^7$ (3)

(c) Expand $(x + 4y)^5$ and simplify. (3)

Question 3

For this group of questions, assume repetitions are not permitted.

(a) How many 3-digit numbers can be formed from the six digits 1, 3, 5, 7, 9, 0? (a) _____ (3)

(b) How many of these numbers are less that 500? (b) _____ (3)

(c) How many are even? (c) _____ (3)

(d) How many are odd? (d) _____ (3)

(e) How many are multiples of 5? (e) _____ (3)

Question 4

A team consists of four boys and six girls. Find the number of ways they can sit in a row if:

(a) The boys and girls are each to sit together (all girls together, all boys together) (3)

(b) Just the girls are to sit together. (3)

Question 5

(a) How many different signals, each consisting of six flags hung in a vertical line, can be formed from four identical red flags and two identical blue flags? (3)

(b) Find the number m of permutations that can be formed from all the letters of the word HITCHHIKER. (3)

Question 6

(a) Find the number m of committees of three that can be formed from eight people. (3)

(b) There are twelve students who are eligible to attend the National Student Association annual meeting. Find the number m of ways a delegation of four students can be selected from the twelve eligible students. (3)

(c) A bag contains 5 red marbles and 6 white marbles. Find the number m of ways that four marbles can be drawn from the bag. (3)

PROBLEMS

Question 7

Assume a quiz has three questions, and:

- the first problem has four true/false questions,
- the second problem requires choosing one of four alternatives, and
- the answer to the third problem is an integer ≥ 15 and ≤ 20

How many different ways is it possible to answer the quiz? (10)

Question 8

How many total functions are there from set A to set B if =$|A| = 3$ and $|B| = 7$? A function f is a total function when the domain of definition of f equals A, the domain of f. (The function f is defined for all inputs). (15)

Question 9

A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters-letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\mathcal{A} = \{A, B, C, \ldots, Z\}$$
$$\mathcal{D} = \{0, 1, 3, \ldots, 9\}$$

using unions $(\cup)$ and set products $(\times)$. (5)

(b) Compute $|L|$, the number of different license plates, using the sum and product rules. (5)

Question 10

Your class tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your Teaching Assistant (TA) has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is $ABCDEFGHIJKL$, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k? (5)

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\}$$

this mapping is j-to-1 for what j? (5)

(c) How many group assignments are possible? (5)

(d) In how many ways can $3n$ students be broken up into n groups of 3? (5)

Question 11

Suppose you have seven dice— each a different color of the rainbow; otherwise the dice are standard, with faces numbered 1 to 6. A *roll* is a sequence specifying a value for each die in rainbow (ROYGBIV) order. For example, one roll is (3, 1, 6, 4, 5, 2)indicating that the red die showed a 3, the orange die showed 1, the yellow 6, …

For the problems below, describe a bijection between the specified set of rolls and another set that is easily counted using the Product, Generalized Product, and similar rules (See Rosen, Chapter 6.3, Definition 1). Then write a simple arithmetic formula, possibly involving factorials and binomial coefficients, for the size of the set of rolls. You do not need to prove that the correspondence between sets you describe is a bijection, and you do not need to simplify the expression you come up with.

For example, let A be the set of rolls where 4 dice come up showing the same number, and the other 3 dice also come up the same, but with a different number. Let R be the set of seven rainbow colors and $S = [1..6]$ be the set of dice values.

Define $B = P_{S,2} \times R_3$, where $P_{S,2}$ is the set of 2-permutations of S and R_3 is the set of size-3 subsets of R. Then define a bijection from A to B by mapping a roll in A to the sequence in B whose first element is a pair consisting of the number that came up three times followed by the number that came up four times, and whose second element is the set of colors of the three matching dice.

For example, the roll

$$(4, 4, 2, 2, 4, 2, 4) \in A$$

maps to

$$((2, 4), \{\text{yellow, green, indigo}\}) \in B$$

Because it is a bijection, $|A| = |B|$, and by the Generalized Product and Subset rules,

$$|B| = 6 \cdot 5 \cdot \binom{7}{3}$$

(a) For how many rolls do exactly two dice have the value 6 and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula.

Example: (6, 2, 6, 1, 3, 4, 5) is a roll of this type, but (1, 1, 2, 6, 3, 4, 5) and (6, 6, 1, 2, 4, 3, 4) are not. (10)

(b) For how many rolls do two dice have the same value and the remaining five dice all have different values? Remember to describe a bijection and write a simple arithmetic formula. (10)
Example: (4, 2, 4, 1, 3, 6, 5) is a roll of this type, but (1, 1, 2, 6, 1, 4, 5) and (6, 6, 1, 2, 4, 3, 4) are not.

(c) **BONUS:** For how many rolls do two dice have one value, two different dice have a second value, and the remaining three dice a third value? Remember to describe a bijection and write a simple arithmetic formula. (10)
Example: (6, 1, 2, 1, 2, 6, 6) is a roll of this type, but (4, 4, 4, 4, 1, 3, 5) and (5, 5, 5, 5, 6, 6, 1, 2) are not.

PROGRAMMING

Question 12 (10)

The Krikkit Bank issues coins in 4 denominations: $1\mathcal{K}, 3\mathcal{K}, 12\mathcal{K}$ and $21\mathcal{K}$.

Given a limited supply of each of the above denominations, in how many ways can you sum them up to a total of $n\mathcal{K}$?

For example, we might input 50 and $\{20, 1, 5, 5\}$. This means we have 20 $1\mathcal{K}$ bills, 1 $3\mathcal{K}$ bill, 5 $12\mathcal{K}$ bills, and 5 $21\mathcal{K}$ bills. How many ways can we use those bills to produce $50\mathcal{K}$?

Please do not use any external libraries.

SUBMISSION DETAILS

Things to submit:

- Submit the following on Blackboard for Assignment 8:
  - The written parts of this assignment as a .pdf named "CS5002_[lastname]_A8.pdf". For example, Ben Bitdiddle's file would be named "CS5002_Bitdiddle_A8.pdf". (There should be no brackets around your name).
  - Make sure your name is in the document as well (e.g., written on the top of the first page).