A4: SETS AND MATRICES

I've always been at the intersection of computers and whatever they can revolutionize.
—Jeff Bezos

Course: CS 5002
Fall 2018
Due: 07 Oct 2018, Midnight

OBJECTIVES

After you complete this assignment, you will be comfortable with:

- Set Notation
- Set operations
- Matrix operations

RELEVANT READING

Rosen:

- Chapter 2.1: Sets
- 2.2: Set Operations
- 2.5: Cardinality of Sets
- 2.6: Matrices

NEXT WEEK'S READING

- 4.1: Divisibility and Modular Arithmetic
- 4.2: Integer Representations and Algorithms,
- 4.3: Primes and Greatest Common Divisors
- 4.4: Solving Congruences

EXERCISES

Problem 1: Rewrite

Rewrite the following statements using set notation:

(a) (1/2 point) A is a subset of C.

(a) ______________________

(b) (1/2 point) The element 3 is not a member of A.

(b) ______________________

(c) (1/2 point) F contains all the elements of G.

(c) ______________________

(d) (1/2 point) A is not a subset of D.

(d) ______________________

(e) (1/2 point) The element 42 is a member of B.

(e) ______________________

(f) (1/2 point) S and T contain the same elements.

(f) ______________________

Points: ______________ out of 3

Problem 2: List the set List

the elements of the following sets.

(a) (½ point) $A = \{x : x \in \mathbb{N}, 6 < x < 8\}$

(a) ______________________________

(b) (½ point) $B = \{x : x \in \mathbb{N}, x \text{ is even}, x < 15\}$

(b) ______________________________

(c) (½ point) $C = \{x : x \in \mathbb{N}, 15 + x = 12\}$

(c) ______________________________

(d) (½ point) $D = \{x : x \text{ is a vowel}, x \text{ is not "a" or "i"}\}$

(d) ______________________________

Problem 3: Set Equality

(a) (½ point) Which of these sets are equal? $\{r, s, t\}, \{t, s, r\}, \{s, r, t\}, \{t, r, s\}$?

(a) ______________________________

(b) (1 point) Consider the following sets:

$$\{9, 3\}$$
$$\{x : x^2 + 12x + 27 = 0\}$$
$$\{x : x \in \mathbb{N}, x \text{ is odd}, 2 < x < 10, x \bmod 3 = 0\}$$

Which of them are equal to $B = \{3, 9\}$?

(b) ______________________________

Problem 4: Subset or not?

The next questions refer to the following sets:

$$\emptyset$$
$$A = \{2\}$$
$$B = \{2, 4\}$$
$$C = \{2, 10, 8\}$$
$$D = \{1, 2, 3, 8, 10\}$$
$$E = \{2, 4, 6, 8, 10\}$$
$$\mathbb{U} = \mathbb{N}$$

Insert the correct symbol $\subseteq$ or $\not\subseteq$. Provide a short explanation.

(a) $\emptyset$ ______________ A

(b) A ______________ B

(c) B ______________ C

(d) B ______________ E

(e) C ______________ D

(f) D ______________ $\mathbb{U}$

Problem 5: Set Operations

$$A = \{1, 2, 3, 4\}$$
$$B = \{3, 5, 7, 9\}$$
$$C = \{3, 4, 5, 6\}$$
$$\mathbb{U} = \{1, 2, 3, ...8, 9\}$$

(a) (½ point) $A \cup B$

(a) ______________________

(b) (½ point) $A \cup C$

(b) ______________________

(c) (½ point) $B \cup B$

(c) ______________________

(d) (½ point) $(A \cup B) \cup C$

(d) ______________________

(e) (½ point) A^c

(e) ______________________

(f) (½ point) $A \setminus B$

(f) ______________________

(g) (½ point) $A \cap (B \cup C)$

(g) ______________________

Problem 6: Power sets

Consider the set:

$A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$

Determine which of the following are true or false:

(a) (½ point) $1 \in A$

(a) ______________________

(b) (½ point) $\{1, 23\} \subseteq A$

(b) ______________________

(c) (½ point) $\{6, 7, 8\} \in A$

(c) ______________________

(d) (½ point) $\{\{4, 5\}\} \subseteq A$

(d) ______________________

(e) (½ point) $\emptyset \notin A$

(e) ______________________

(f) (½ point) $\emptyset \subseteq A$

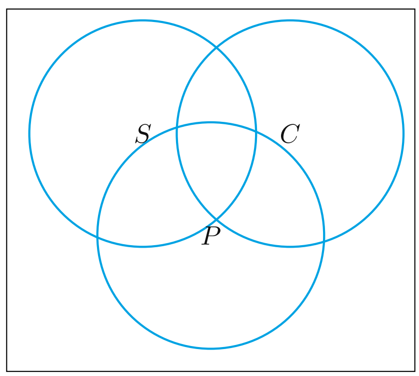
(f) ______________________

Problem 7: Pokemon: Catch 'em all

In a survey of 75 Pokemon Go players, it was found that 35 of them have captured Skipbloom; 40 have caught Charmander; and 25 had caught Pikachu. Further, 15 had both Skipbloom and Charmander; 4 had both Skipbloom and Pikachu; and 19 of them had both Charmander and Pikachu. 10 unfortunate Pokemon Go players had not caught any of these three Pokemon.

(a) (2 points) How many people caught all three Pokemon?

(b) (4 points) Fill in the correct number of people in each of the eight regions of the Venn diagram. S, C, and P denote Skipbloom, Charmander and Pikachu respectively.



(c) (2 points) Determine the number of people who caught exactly one Pokemon. Show your work for full credit.

Problem 8: Vectors

In class we talked about matrices; vectors are simply a 1-dimensional matrix. A Vector can be either a row vector ($1 \times n$ matrix) or a column vector (a $n \times 1$ matrix). We can apply addition, scalar multiplication, and negation to vectors. *Note: only vectors of the same length/shape can be added, just as in matrix addition.*

Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, and $w = (0, 5, -8)$

(a) (1 point) Find $u + v$.

(a) ______________________

(b) (1 point) Find $v + w$.

(b) ______________________________

(c) (1 point) Find $-3u$

(c) ______________________________

(d) (1 point) Find $-w$

(d) ______________________________

(e) (1 point) Find $3u - 4v$. (The order of operations is to do the scalar multiplication first, then the addition).

(e) ______________________________

Problem 9: Column Vectors

The previous problem showed row vectors; column vectors are vertical:

$$\begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}3\\-7\end{pmatrix}, \begin{pmatrix}2\\3\\15\\0\end{pmatrix}$$

Compute the following:

(a) (1 point)

$$\begin{pmatrix}3\\-4\\5\end{pmatrix} + \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

(a) ______________________________

(b) (1 point)

$$\begin{pmatrix}1\\2\\-3\end{pmatrix} + \begin{pmatrix}4\\-5\end{pmatrix}$$

(b) ______________________________

(c) (1 point)

$$-3 \cdot \begin{pmatrix}4\\-5\\-6\end{pmatrix}$$

(c) ______________________________

$$4\begin{pmatrix}2\\5\\7\end{pmatrix} + 3\begin{pmatrix}9\\-2\\4\end{pmatrix}$$

(c) ______________________________

Problem 10: Matrix Operations

(a) (1 point)

$$\begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ -5 & 8 \\ 2 & 1 \end{bmatrix}$$

(b) (1 point)

$$4 \cdot \begin{bmatrix} 1 & 6 \\ 2 & 4 \\ -3 & 7 \end{bmatrix}$$

Problem 11: Transpose

We mentioned transpose in class, but didn't have time to do an example. To transpose a matrix, we write the columns as rows and vice versa:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Find S^T below:

(a) (1 point)

$$S = \begin{bmatrix} 1 & 3 & 5 \\ 6 & -7 & -8 \end{bmatrix}$$

(a) ______________________

(b) (1 point)

$$S = \big(5, 2, 1\big)$$

(b) ______________________

Problem 12: Transpose

Show that $(A + B)^T = A^T + B^T$.

Problem 13: Crossing Sets

Let $A = \{a, b, c\}$, $B = \{x, y\}$, and $C = \{0, 1\}$. Find

(a) (1 point) $A \times B$

(a) ______________________

(b) (1 point) $C \times B$

(b) ______________________

(c) (1 point) $B \times A$

(c) ______________________

(d) (1 point) $B \times C \times B$

(d) ______________________

Problem 14: Supersets

Prove : For any set A, $|A| < |\mathcal{P}(A)|$ where $\mathcal{P}(A)$ is the powerset of A.

PROBLEMS

Problem 15: Set Formulas and Propositional Formulas

(a) (2 points) Verify that the propositional formula $(P \wedge \neg Q) \vee (P \wedge Q)$ is equivalent to P.

(b) (4 points) Prove that $A = (A - B) \cup (A \cap B)$ for all sets, A, B by showing $x \in A iff x \in (A - B) \cup (A \cap B)$ for all elements x using the equivalence of part (a) in a chain of IFF's.

Problem 16: Subset Take-away

Subset take-away is a two player game played with a finite set A of numbers. Players alternately choose nonempty subsets of A with the conditions that a player may not choose the whole set A, or any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if the size of A is one, then there are no legal moves and the second player wins. If A has exactly two elements, then the only legal moves are the two one-element subsets of A. Each is a good reply to the other, and so once again the second player wins.

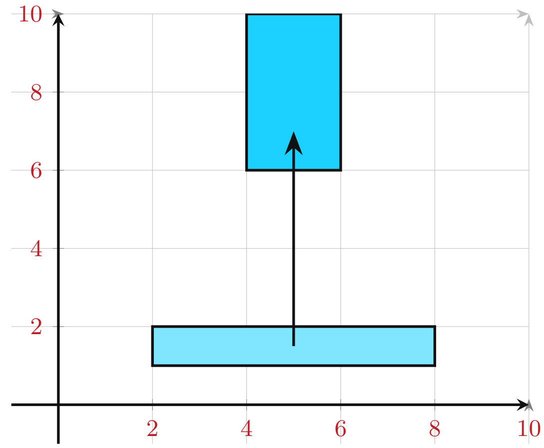
The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element.

In both cases, these moves lead to a situation that is the same as the start of a game on a set with two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.

Problem 17: Compound Transforms

Below is a diagram of a polygon (rectangle) that undergoes a transformation.



(a) (3 points) Define the compound transform (transformation matrix) that will transform the original polygon to the final polygon.

(b) (4 points) Apply the transform to the 4 corners of the original polygon. Show your work, and confirm that each point lines up with the diagram.

Problem 18: Transformation Matrices

Prove that a multiplication of transformation matrices for two successive scalings is commutative (that is, if we have scaling matrix S_1 and S_2, prove that $S_1 \times S_2 = S_2 \times S_1$).

Problem 19: Sets, Functions and Politics

The language of sets and relations may seem remote from the practical world of programming, but in fact there is a close connection to relational databases, a very popular software application building block implemented by such software packages as MySQL. This problem explores the connection by considering how to manipulate and analyze a large data set using functions and operators over sets. Systems like MySQL are able to execute very similar high-level instructions efficiently on standard computer hardware, which helps programmers focus on high-level design.

Below, we describe a fairly straightforward *data model*. This data model represents citizens in a country, the politicians, votes, and a bill. In this country, citizens live in a region. Each citizen can vote for one senator; assume that in this scenario, each citizen has voted for one of the senators (all senators that got a vote became a senator). Each senator is voting for a bill, and the vote is either Aye, Nay or Abstain.

We describe this with the following constructs:

- A set C of citizens
- A set R of regions the citizens live in
- A set S of senators
- A set T of votes to *tally* (aye/nay/abstain) for the Bill
- A function $L(c, r)$: Citizen $c \in C$ lives in region $r \in R$
- A function $V(c, s)$: Citizen $c \in C$ voted for senator $s \in S$
- A function $B(s, t)$: Senator $s \in S$ submits a ballot with a tally $t \in T$.

This problem asks you to produce an informal, intuitive query over our data, using standard set and functional operators, such that the expression you provide can be interpreted as answering the query.

- $\cup$ (union)
- $\cap$ (intersection)
- $\setminus$ (set difference)
- $|S|$ (cardinality)
- $f^{-1}[S] = \{x \in X | f(x) \in S\}$: Function pre-image: The *set* that is the preimage of set S. Note: this is very similar to the *inverse function*, but the image is a set, not a function; since we're not working with 1:1 functions, the inverse function won't apply. Can also use $f^{-1}[\{s\}] = \{x \in X | f(x) = s \in S\}$ (that is, s is a specific element of S, not the entire image).
- $(R \circ S)(a, c)$ (That is, $\exists b \in B$ s.t. $S(a, b) \wedge R(b, c)$. That is, a in S is related to c in $R \circ S$ if starting at a, you can follow an S arrow to the start of an R arrow and then follow the R arrow to get to c.

Here's an example:

Search query: Find the set of senators who voted "Aye" for the bill.

Answer: $B^{-1}[\{\text{"Aye"}\}]$

(a) (2 points) Draw a Venn-like diagram to show the data model (like we've used to show function mappings before).

(b) (2 points) Describe $|C|$.

(c) (2 points) What does it mean if $|V| \neq |C|$?

(d) (2 points) The set of citizens who voted for senator "Mackenzie Allen".

(e) (2 points) How many citizens of "Springfield" voted for senator "Barbara Gordon"?

(f) (2 points) The set of senators voted for by citizens in the "Stepford" region.

(g) (2 points) The function that relates the citizen c with the vote "Aye".

(h) (2 points) The set of senators that abstained.

(i) (3 points) We've asked you to specify some sets and functions that represent specific queries. List 3 other queries you might want to execute on this data set.

PROGRAMMING

In this section, we provide a couple of problems to help you start connecting the content of this class to programming.

Problem 20: Python and Matrix Multiplication

Write a Python function that takes in two parameters: The first is a 1x3 array that represents an (x, y) coordinate. The second is a 3x3 array that is a composite transformation matrix. Your function should compute the transformation and return a new 1x3 array with the new coordinates.

For full credit, provide your code, and ensure that it is readable. We should be convinced that it works by reading it, without necessarily running it.

Problem 21: Python and Sets

Write a function that takes in a set and lists out all the subsets for the provided set. Once again, your code should be readable and clearly correct without running it.

SUBMISSION DETAILS

Things to submit:

- Submit the following on Blackboard for Assignment 4:
  - The written parts of this assignment as a .pdf named "CS5002_[lastname]_A4.pdf". For example, my file would be named "CS5002_Slaughter_A4.pdf". (There should be no brackets around your name).
  - Make sure your name is in the document as well (e.g., written on the top of the first page).