A3: SUMS AND SEQUENCES

The sum of wisdom is that time is never lost that is devoted to work.
– Ralph Waldo Emerson

Course: CS 5002
Fall 2018
Due: 30 Sept 2018, Midnight

OBJECTIVES

After you complete this assignment, you will be comfortable with:

- Define a sequence and a summation.
- Interpret and utilize summation notation.
- Recognize countable and uncountable summations or sequences.
- Manipulating summations
- Evaluating summations
- Recognizing and applying summation formulae
- Handling factorials

READING (THIS WEEK)

- Rosen, Ch 2.4 (Sequences and Summations).
- Optional: Discrete Math: Chapter 2 (Intro), 2.1 (Definitions), 2.2 (Arithmetic and Geometric Sequences) (http://discrete.openmathbooks.org/dmoi/ch_logic.html)

READING (NEXT WEEK)

- Rosen, Chapter 2
- - 2.1 (Sets)
  - 2.2 (Set Operations)
  - 2.5 (Cardinality of Sets)
  - Matrices
- Optional: Discrete Math: Chapter 0.3 (Sets) (http://discrete.openmathbooks.org/dmoi/ch_logic.html)

NOTATION

Symbol	Meaning
$\{a_1, \ldots, a_n\}$	
$\{x \vert P(x)\}$	
$\mathbb{Z}$	is all integers (e.g. -2, -1, 0, 1, 2, 3, ...)
$\mathbb{N}$	is all natural numbers (e.g. 1, 2, 3, 4, ...)
$\mathbb{R}$	is all real numbers (e.g. -0.5, -0.6, ... 0, ... 1.111, 1.1112, ...)
$\mathbb{Z}^+$	is non-negative integers (e.g. $0, 1, 2, \ldots$)
$\mathbb{Q}$	is all rational number (e.g. 1/1, 1/2, 2/1, 2/2, 2/3, ...– anything that is a numerator over a non-zero denominator.
$\mathbb{R} > \mathbb{Q} > \mathbb{Z} > \mathbb{N}$	($\mathbb{R}$ contains all numbers in $\mathbb{Q}$, which contains all numbers in $\mathbb{Z}$, which contains all $\mathbb{N}$, the natural numbers).
$S = T$	set equality
$\emptyset$	empty (null) set
$x \in S$	x is a member of S

Symbol	Meaning
$x \notin X$	x is not a member of S
$S \subseteq T$	S is a subset of T
$S \subset T$	S is a proper subset of T
$\vert S \vert$	cardinality of S
$P(S)$	power set of S
(a, b)	ordered pair
$(a_1, \ldots, a_n)$	n-tuple
$A \times B$	Cartesian product of A and B
$A \cup B$	A union B
$A \cap B$	A intersect B
$A - B$	the difference of A and B
$\overline{A}$	complement of A
$\cup_{i=1}^{n} A_i$	union of $A_i, i = 1, 2, \ldots n$
$A \oplus B$	symmetric difference of A and B

 Points: ____________ out of 0

EXERCISES

1. What is the a_8 term of the sequence a_n if a_n equals:

(1) (a) 3^{n-1}

(a) ______________

(1) (b) 3

(b) ______________

(1) (c) $n - 1(-1)^n$

(c) ______________

(1) (d) $-(-2)^n$

(d) ______________

2. What are the terms a_0, a_1, a_2 of the sequence a_n, where a_n equals

(1) (a) $(-2)^n$

(a) ______________

(1) (b) 4

(b) ______________

(1) (c) $3 + 2^n$

(c) ______________

(1) (d) $2^n + (-2)^n$

(d) ______________

3. List the first 6 terms of each of these sequences:

(1) (a) The sequence starting with 25 and obtaining each term by subtracting 2 from the previous term.

(a) ______________________________

(1) (b) The sequence starting with $a_1 = 1$ whose nth term is the sum of the first $n - 1$ positive integers.

(b) ______________________________

(1) (c) The sequence whose first two terms are 1 and 4, and each succeeding term is the sum of the previous two terms.

(c) ______________________________

4. Find the first six terms of the sequence defined by each of the recurrence relations and initial conditions.

(1) (a) $a_n = -2a_{n-1}, a_0 = -1$

(a) ______________________________

(1) (b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$

(b) ______________________________

(1) (c) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$ (For this sequence, find 8 terms)

(c) ______________________________

5. Show that the sequence a_n is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

(1) (a) $a_n = 0$

(1) (b) $a_n = 1$

(1) (c) $a_n = (-4)^n$

6. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that shown in class. Show your work, with at least 3 substitutions. You may choose to iterate forward or backward.

(2) (a) $a_n = -a_{n-1}, a_0 = 3$

(b) $a_n = a_{n-1} + 4, a_0 = 1$

(c) $a_n = a_{n-1} - n, a_0 = 3$

7. What are the values of these sums, where $S = \{2, 3, 8, 12\}$?

(1) (a) $\sum_{j \in S} j$

(a) ________________

(1) (b) $\sum_{j \in S} j^2$

(b) ________________

(1) (c) $\sum_{j \in S} (1/j)$

(c) ________________

(1) (d) $\sum_{j \in S} 1$

(d) ________________

8. Find the value of each of these sums.

(1) (a) $\sum_{j=0}^{3} 3 \cdot (1 + (-1)^{j-1})$

(1) (b) $\sum_{j=0}^{2} (j^3 - 2^j)$

9. Compute each of these double sums

(2) (a) $\sum_{i=1}^{3} \sum_{j=1}^{2} (2i + j)$

(2) (b) $\sum_{i=1}^{3} \sum_{j=0}^{2} j$

(2) (c) $\sum_{i=0}^{2} \sum_{j=1}^{3} i^3 j^2$

10. Simplify the following factorial expressions, showing your work.

(2) (a)

$$\frac{(n+1)!}{n!}$$

(2) (b)

$$\frac{n!}{(n-2)!}$$

(2) (c)

$$\frac{(n-r+1)!}{(n-r-1)!}$$

PROBLEMS

11. A person deposits \$1000 in an account that yields 9% interest compounded annually.
Compounded annually means that at the end of each year (annually = once per year), the interest is calculated based on the amount in the account at that point in time, and then added to the account.

(3) (a) Set up a recurrence relation for the amount in the account at the end of n years.

(3) (b) Find solution to the recurrence for the amount in the account at the end of n years.

(2) (c) How much money will the account contain after 100 years?

12. Sammy the Shark is a financial service provider who offers loans on the following terms: Sammy loans a client m dollars in the morning. This puts the client m dollars in debt to Sammy.

Each evening, Sammy first charges a service fee which increases the client's debt by f dollars, and then Sammy charges interest, which multiplies the debt by a factor of p.

For example, Sammy might charge a "modest" ten cent service fee and 1% interest rate per day, and then f would be $\$0.10$ and p would be 0.01%.

(a) What is the client's debt at the end of the first day? Provide a formula in terms of m, f and p.

(b) What is the client's debt at the end of the second day? Provide your answer as a formula in terms of m, f, p.

(c) What is a client's debt at the end of the second day if the client borrowed $\$10$, the fee $f = \$0.10$, and the rate is 1%.

(d) Write a formula for the client's debt after d days, and find an equivalent closed form. Show your work.

(2) (c) If you borrowed $\$10$ from Sammy for a year (assume 365 days), how much would you owe him? The fee $f = 0.10$, and the rate is 1%.

13. In class, we evaluated how much you would have earned in 5 years if you had a base salary of $100,000$ and a 5% raise each year. Your friend Sally negotiated the following deal with another company: A starting salary of $105,000$, and an annual raise of $\$5,000$.

(2) (a) Set up a recurrence relation for Sally's salary n years after starting.

(2) (b) Find an explicit formula for the salary n years after after starting.

(1) (c) How much will Sally earn in her 5^{th} year at this company?

(d) How will Sally have earned in 5 years? (Show your work, and provide a closed form for the summation)

(2) (e) Who is going to have a higher salary after 10 years in the same job?

(2) (f) Who is going to have made more, cumulatively, in those 10 years?

(5) 14. What's wrong with the following derivation?

$$\begin{aligned}\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{k=1}^{n} \frac{1}{a_k}\right) &= \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{a_j}{a_k} \\ &= \sum_{k=1}^{n}\sum_{k=1}^{n} \frac{a_k}{a_k} \\ &= \sum_{k=1}^{n} n \\ &= n^2\end{aligned}$$

15. We begin with two large glasses. The first glass contains a pint of water, and the second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass and repeat this pouring back-and-forth process a total of n times.

(5) (a) Describe a closed-form formula for the amount of wine in the first glass after n back-and-forth pourings.

(5) (b) What is the limit of the amount of wine in each glass as n approaches infinity?

16. I like to ride my bike really long distances in the middle of nowhere. I've found that I can ride really long distances, as long as I have enough food and water. But, I can only carry so much food and water on my bike. Food isn't a problem, but water is heavy and big. In fact, I can only carry 1 gallon of water, and that's enough for 1 day of riding.

Similar to the sport of ultramarathoning, I like to see how far out into the middle of nowhere I can go. I also like to do it entirely self-sufficiently: no support staff. I can't assume that there is a place to get drinkable water along the way—this is the middle of nowhere. That means I can only go as far as I can carry water. I need water at all times, but I realized I could go further if I place caches of water along the way.

Let's say that if I start at the beginning with 1 gallon of water and ride straight out, I can ride r miles until I run out of water. Then, of course, I'm stuck in the middle of nowhere with no water, so I'll die. (well, let's hope not, but it won't be comfortable).

I could easily ride to a point 2/3 of r miles out in two days, by doing something like this: Leave the starting point with 1 gallon of water. Ride $r/3$ miles into nowhere, cache 1/3 of my water, and then return back to the starting point where I sleep for the night. The next day, I can ride to the cache, top off my water (which gives me a full gallon at that point), ride another $r/3$ miles (at which point I have 2/3 of a gallon of water, and am $2r/3$ miles out), then turn around and get back to the starting point by the time I run out of water.

But what if I wanted to go further?

(3) (a) What is the furthest I can ride and return to my starting point, with no water pre-cached, if I have only 1 gallon when I leave the start?

(4) (b) Let's say that at my starting point, I only have 2 gallons of water. How far out is the farthest point I can reach without running out of water by the time I get back to my starting point? (I can still only carry 1 gallon at a time, but I can do some caching).

(6) I've come up with a recursive strategy to go really far into nowhere and then return, using n gallons of water. I plan to build up a cache of $n-1$ gallons plus enough to get back home some fraction of a day's ride into nowhere. On the last cache delivery, rather than returning to the start, I continue to recursively cache water further into nowhere. Then, the cache has just enough water left to get me home.
Prove that with n gallons of water, this strategy will get me $H_n/2$ days into nowhere and back, where H_n is the n^{th} Harmonic number:

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$$

Use this to show that I can go as far as I want from the starting point.

BONUS

As we've noted, this class has a wide range of abilities. In that spirit, I offer this challenge problem for you to help you push your understanding of the content a bit further.

These questions should not be attempted before the rest of the assignment.

These questions do not contribute to your grade, but can count toward participation points, ONLY IF the rest of the assignment is completed.

17. Cribbage players are extremely aware that $15 = 8+7 = 4+5+6 = 1+2+3+4+5$. Find the number of ways to represent 1050 as a sum of consecutive positive integers. (The trivial representation of '1050' is one way; there are 4 ways to represent 15 as a sum of consecutive integers. You do not need to know anything about cribbage to solve this.)

Page	Points	Score
2	14	
3	5	
4	8	
5	8	
6	6	
7	8	
8	8	
9	7	
10	13	
11	10	
12	7	
13	6	
Total:	100	

SUBMISSION DETAILS

Submit your answers to the math problems on Blackboard as a pdf. You can do this by printing out this assignment, hand-writing your answers, and then scanning in, or by using a program such as Preview on the Mac to annotate the pdf. Instructions on how to scan your document using the scanner/copiers on campus are provided on the course website ().
Things to submit:

- Submit the following on Blackboard for Assignment 3:
  - The written parts of this assignment as a .pdf named "CS5002_[lastname]_A3.pdf". For example, my file would be named "CS5002_Slaughter_A3.pdf". (There should be no brackets around your name).
  - Make sure your name is in the document as well (e.g., written on the top of the first page).