# A11: PROVING ALGORITHM CORRECTNESS. GRAPHS AND TREES

**You can graph human evolution, which is mostly a straight line, but we do get better and change over time, and you can graph technological evolution, which is a line that's going straight up. They are going to intersect each other at some point, and that's happening now. —Daniel H. Wilson**

**Course: CS 5002**
**Fall 2018**
**Due: Dec 7, 2018, Midnight**

## OBJECTIVES

After you complete this assignment, you will be comfortable with:

- Several proof techniques, including:
  - Proof by counterexample
  - Proof by induction
  - Proof by loop invariance
- Some basic tree terminology
- Some special trees
- Tree traversals
- Some basic graph terminology
- Graph representations

## RELEVANT READING

Rosen:

- Chapter 5.1. Mathematical induction
- Chapter 5.2 Strong Induction and Well-Ordering
- Chapter 5.3 Recursive Definitions and Structural Induction
- Chapter 5.5. Program Correctness
- Chapter 11.1 Introduction to Trees
- Chapter 11.3 Tree Traversals
- Chapter 10.1 Graphs and Graph Models
- Chapter 10.2 Graph Terminology and Special Types of Graphs

## NEXT WEEK'S READING

Rosen,

- Chapter 7: Discrete Probability

## EXERCISES

**Question 1**

Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for a positive integer $n$.

(a) What is the statement $P(1)$?

(b) Show that $P(1)$ is true, completing the basis step of the proof.

(c) What is the inductive hypothesis?

(d) What do you need to prove in the inductive step?

(e) Complete the inductive step, identifying where you use the inductive hypothesis.



Points: ______________ out of 6

## Question 2

Briefly explain what is wrong with the following "proof":

*Theorem:* For every positive integer $n$, if $x$ and $y$ are positive integers with max$(x, y)$ = n, then $x = y$.

*Basis step:* Suppose that $n = 1$. If max$(x, y)$ = 1, and $x$ and $y$ are positive integers, then we have $x = 1$ and $y = 1$.

*Inductive step*: Let $k$ be a positive integer. Assume that whenever max$(x, y) = k$ and $x$ and $y$ are positive integers, then $x = y$. Now let max$(x, y) = k + 1$, where $x$ and $y$ are positive integers. Then max$(x - 1, y - 1) = k$, so by inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

## Question 3

Prove that for every positive integer $n$:

$$1 \cdot 2 + 2 \cdot 3 + \cdot + n(n + 1) = n(n + 1)(n + 2)/3$$

**Question 4**

Give a recursive algorithm for computing $nx$ whenever $n$ is a positive integer, and $x$ is an integer, using just addition. Please provide pseudocode fro your algorithm.

**Question 5**

Use merge sort to sort $b, d, a, f, g, h, z, p, o, k$ into increasing order. Show all steps used by the algorithm.

**Question 6**

Use quick sort to sort 3, 5, 7, 8, 1, 9, 2, 4, 6 into increasing order. Show all steps used by the algorithm.

**Question 7**

(a) Give a recursive definition of the length of a string.

(b) Use the recursive definition from part (a) to prove that, given two strings $x$ and $y$, it holds that $l(xy) = l(x) + l(y)$, where $l(\cdot)$ denotes the length of a string.

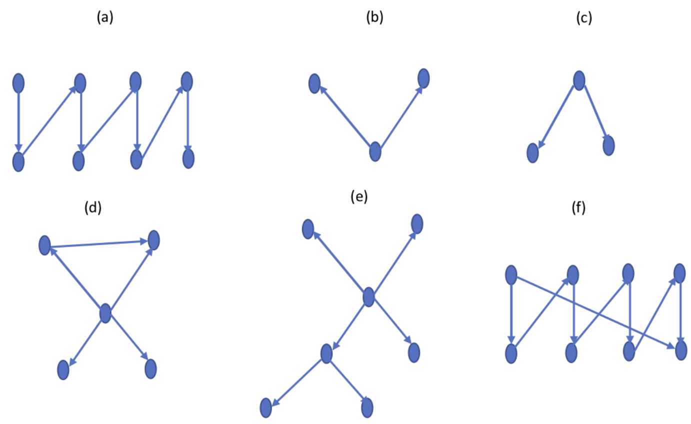


Figure 1: Graphical representations of trees, used in Question 8.

**Question 8**

Consider the following graphs, represented in Figure 1. Which of the presented graphs are trees? If you think that some of the graphs (a)–(f) are not trees, please explain why.

 Points: ______________ out of 3

**Question 9**

Draw graphs that have the following adjacency matrices:

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Find adjacency lists that correspond to the adjacency matrices from parts (a) and (b).

**Question 10**

(a) Represent the following expressions using binary trees, such that operands are leaves and operations are internal nodes:
- $(x + xy) + (x/y)$
- $x + ((xy + x)/y)$

Now, traverse the constructed trees using the following traversals:

(b) Pre-order traversal.

(c) In-order traversal.

(d) Post-order traversal.

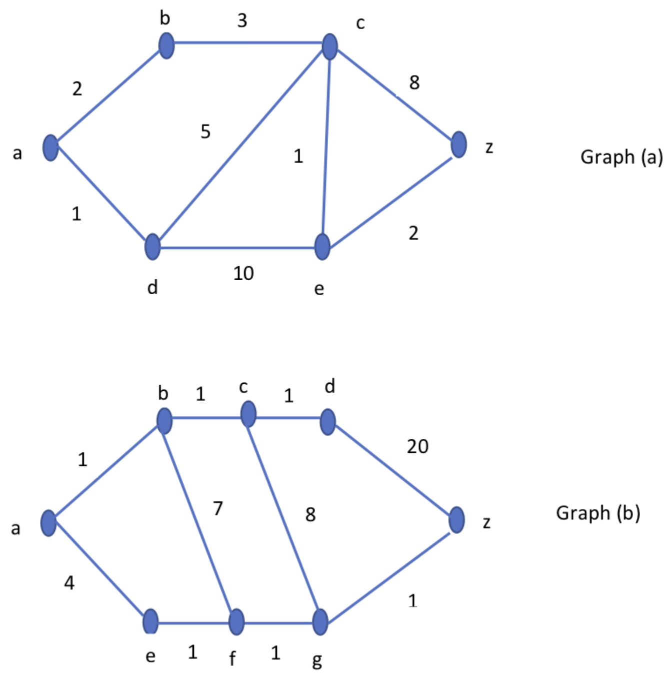


Figure 2: Graphical representations of graphs, used in Question 11.

**Question 11**

Use Dijkstra's algorithm to find the shortest path from $a$ to $z$ for graphs represented in Figure 2. For both graphs, list all steps of the algorithm.

 Points: __________ out of 6

## PROBLEMS

**Question 12**

Using mathematical induction, prove that 2 divides $n^2 + n$ whenever $n$ is a positive integer.

**Question 13**

Prove that $f_1 + f_3 + f_3 + \cdots + f_{2n-1} = f_{2n}$.

### Question 14

A chain letter starts when a person sends a letter to five others. Each person who receives the letter either sends it to five other people who have never received it, or does not send it to anyone. Suppose that 10000 people send out the letter before the chain ends, and that no one receives more than one letter. How many people receive the letter, and how many do not send it out?

### Question 15

Prove that if some adjacency matrix $A$ is an $m \times m$ symmetric matrix, then $A^2$ is also symmetric.

**Question 16**

Show that a simple graph is a tree if and only if it is connected, but the deletion of any of its edges produces a graph that is not connected.

**Question 17**

Suppose there exists an integer $k$ such that every man on a desert island is willing to marry exactly $k$ of the women on the island, and every woman on the island is willing to marry exactly $k$ of the men. Let's also assume that a man is willing to marry a woman if and only if she is willing to marry him. Show that it is possible to match men and women on the island so that everyone is matched with someone that they are willing to marry.

| Question | Points | Score |
|---|---|---|
| Simple Proof By Induction | 6 | |
| Lame Proof | 4 | |
| Another Proof by Induction | 4 | |
| Simple Recursive Algorithm | 3 | |
| Merge Sort | 2 | |
| Quick Sort | 2 | |
| Strings and Recursion | 5 | |
| Trees | 3 | |
| Adjacency Matrix | 4 | |
| Binary Trees | 10 | |
| Dijkstra's Algorithm | 6 | |
| Mathematical Induction | 6 | |
| Mathematical Induction | 4 | |
| Trees and Proofs | 10 | |
| Adjacency Matrix and Proofs | 10 | |
| Connected Graphs and Treees | 10 | |
| Marriage Problem | 11 | |
| Total: | 100 | |

# SUBMISSION DETAILS

Things to submit:

- Submit the following on Blackboard for Assignment 11:
  - The written parts of this assignment as a .pdf named "CS5002_[lastname]_A11.pdf". For example, Ben Bitdiddle's file would be named "CS5002_Bitdiddle_A11.pdf". (There should be no brackets around your name).
  - Make sure your name is in the document as well (e.g., written on the top of the first page).