CS5002
3/27 - Week 12

Admin
- HW6 was due 6pm
- Exam 2 4/10 in class
  - ↳ practice problems now
    next week - review solutions
- HW7 out, due 4/3 9pm
  last HW!
- 4/24 makeup exam q.

Agenda
1. Inductive reasoning
2. Structure of inductive proof

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3. Induction: inequality
4. structural induction

$\boxed{\text{mathematical induction}}$ Proof technique:

$\boxed{\text{0. Review}}$

Logic Statement - declarative, has truth value

Predicate - generalization of a logic statement

P(x) x knows karate

→ reason prove

→ can't prove

predicate ⇝ logic statement

1. plug in x P(Nora)
2. quantifiers $A = \{s \mid s \text{ is in } \text{Sh } 135\}$

$\forall x \in A \; P(x)$

$\exists x \in A \; P(x)$

$\boxed{\text{1. Inductive Reasoning}}$

ATM to get \$ ⟶ only has \$2, \$5

Carly asks: what amounts can we make?

Starting point: we can make any dollar amount using \$2, \$5

$S(x)$ → any \$$x$

↳ logic statement

$S(4) = 2+2$ ✓ $S(7) = 5+2$ ✓ $S(21) = 3\cdot 5 + 3\cdot 2$ ✓

$S(11) = 3\cdot 2 + 1\cdot 5$ ✓ $S(8) = 4\cdot 2$ ✓

Inductive proof is used as a shortcut for all $S(\dots)$

How do we do the shortcut?

- $S(x)$ for smallest value of x we care about $S(4)$
- If I have $S(k)$, I can get to $S(k+1)$ $S(k) \Rightarrow S(k+1)$

Have \$k in \$2, \$5. want to get to \$k+1

- drop a 5, get 3 2's $(-5, +6)$
- drop two 2s, get a 5 $(-4, +5)$

2. Structure of Inductive Proof

- our job is to be convincing!
- follow the structure
- small steps
- label everything

1. Predicate
2. Logic statement
3. Base case
4. Inductive step $S(k) \Rightarrow S(k+1)$

← apply to \$2/\$5 problem

Predicate $P(x)$ $x = 2\cdot t + 5\cdot f$ for $t, f \in \mathbb{N}$

Logic statement $\forall x \in \mathbb{Z}^+ \; x \geq 4 \Rightarrow P(x)$

Base case
(smallest value we care about)

$P(4) = 2t + 5f$ where $t = 2, f = 0$

Inductive step $P(k) \Rightarrow P(k+1)$

↳ Inductive Hypothesis Assume $P(k)$ $k = 2t + 5f$ $t, f \in \mathbb{N}$

tiny steps to get from k to $k+1$

want: $k+1 = 2t' + 5f'$

$k = 2t + 5f$

$k+1 = 2t + 5f + 1$

case one

$k+1 = 2t + 5f + 1$

$= 2t + 5f + (6-5)$

$= 2t + 6 + 5f - 5$

$= 2(t+3) + 5(f-1)$ ✓

case two

$k+1 = 2t + 5f + 1$

$= 2t + 5f + (5-4)$

$= 2t - 4 + 5f + 5$

$= 2(t-2) + 5(f+1)$ ✓

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Problem #1

In class today, we showed that any monetary value $n where $n > 3$ car
bills. Now we'll extend that idea to a few other cases.

Proofs are not required for these, but include a brief explanation of the
Could any monetary value $n where $n > k$ (you can choose the value c

$5 and $10 bills?

no :'(

$25 and $1 bills?

yes

$4 and $5 bills?

yes for $n \geq \$12$

$3 and $7 bills?

yes for $n \geq \$12$

Problem #2

For one of the options you said "yes" to, above, prove it using mathematical induction. As always with this kind of proof, follow the structure, and show and explain every step. We've started you off with the outline:

\$4, \$5

Predicate

$S(x)\ \ x = 4 \cdot f + 5 \cdot g$ for $f, g \in \mathbb{N}$

Logic Statement

$\forall x \in \mathbb{Z}^+\ \ x \geq 12 \Rightarrow S(x)$

Base Case

$S(12) = 4 \cdot 3$ ✓

Inductive Step $S(k) \Rightarrow S(k+1)$

$\boxed{\text{IH}}$ $S(k)$ is true $k = 4 \cdot f + 5 \cdot g$

get to $S(k+1)$

- add \$5, take away \$4
- add \$16, take away \$15

3. Induction: Inequality

Predicate $P(x)$ $\quad 5x+5 \leq x^2$

Logic Statement $\quad \forall x \in \mathbb{Z}^+\ x \geq 6 \Rightarrow P(x)$

Base case $\quad P(6)\quad 5\cdot 6+5 \leq 6^2$

$35 \leq 36$ ✓

Side notes

$P(1)\quad 5+5 \leq 1^2$ ✗

$P(2)\quad 10+5 \leq 2^2$ ✗

$P(3)\quad 15+5 \leq 3^2$ ✗

$P(4)\quad 20+5 \leq 4^2$ ✗

$P(5)\quad 25+5 \leq 5^2$ ✗

$P(6)\quad 30+5 \leq 6^2$ ✓

$P(7)\quad 35+5 \leq 7^2$ ✓

$P(8)\quad 40+5 \leq 8^2$ ✓

Goal: $P(k+1)$

$5(k+1)+5 \leq (k+1)^2$

Inductive step $\quad P(k) \Rightarrow P(k+1)$

IH $\quad P(k) \quad 5k+5 \leq k^2$

$$
\begin{aligned}
5(k+1)+5 &= 5k+5+5 && \text{// expansion}\\
&\leq k^2+5 && \text{// by IH ***}\\
&\leq k^2+6 \\
&\leq k^2+k+k+1 && \text{// because } k \geq 6\\
&= k^2+2k+1 \\
&= (k+1)^2 && \text{✓ done!}
\end{aligned}
$$

8:01

$5(k+1)+5 \leq (k+1)^2$!!

Goal: $5(k+1)+5 \leq (k+1)^2$

Rewrite:

$5(k+1)+5 = 5k+5+5$

$5k+5+5 \leq (k^2)+5$

$k^2+5 \leq k^2+6$

$k^2+6 \leq k^2+k+k+1$

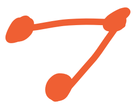
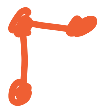
$k^2+k+k+1 = k^2+2k+1 = (k+1)^2$

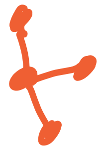
4. Structural Induction ~ set, graph, tree

- show something is true as structure gets bigger
  - add vertex, edge, element
  - property is true no matter how big
- usually, start with k+1 and remove something

Ex: graph example total degree of a graph $2 \cdot |E|$

 1 edge deg 2

 2 edges deg 4  2 edges deg 4

 3 edges deg 6  3 edges deg 6

Predicate $S(x)$ graph with x edges has $2x$ total degrees

Logic Statement $\forall x \in \mathbb{Z}^+ \; S(x)$ $S(0)$

Base case $S(1)$  edge (u,v)

u has deg = 1, v has deg = 1

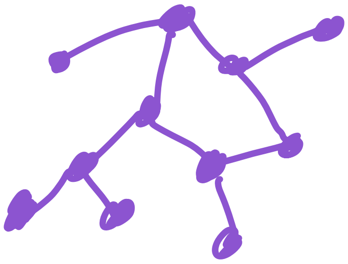
Goal: $S(k+1)$ k+1 edges $2(k+1)$ degree

Inductive Step $S(k) \Rightarrow S(k+1)$

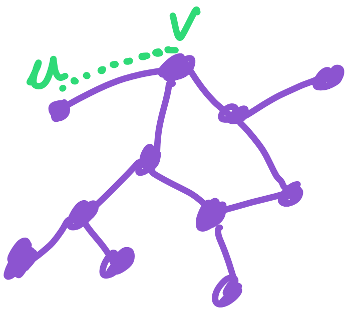
IH $S(k)$ graph with k edges has degree $2k$

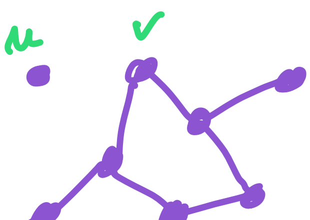
Graph with k+1 edges

(total deg: unknown)

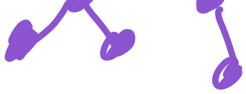


remove edge (u, v)



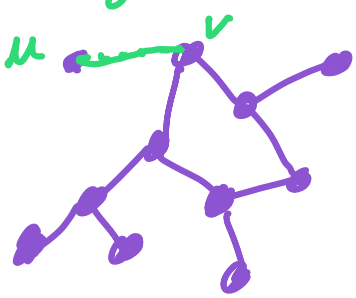


Graph with k edges



total degree $2k$ (by IH)

add edge (u,v) back



- edge u,v is incident on u (one degree) and v (one degree)
- total degree: $\underbrace{2k}_{\text{by IH}} + \underbrace{2}_{u,v} = 2(k+1)$ ✓

ex: sets Set A, $|P(A)| = 2^{|A|}$ $\quad P(A)$ set of all subsets of A

$A = \{a, b, c\}$

$\overline{0/1}\ \overline{0/1}\ \overline{0/1} = 2^3$

Predicate $\quad S(x)$ if $|A| = x$ then $|P(A)| = 2^x$

Logic statement $\quad \forall x \in \mathbb{Z}^+\ S(x)$

Base case $\quad x = 1$ arbitrary element in $A = \{a\}$

$P(A) = \{\{\}, \{a\}\}$ $\quad |P(A)| = 2 = 2^x = 2^1$

Goal: $S(k+1)$ $\quad |A| = k+1$ $\quad |P(A)| = 2^{k+1}$

Inductive Step $\quad S(k) \Rightarrow S(k+1)$

IH $\quad S(k)$ $\quad |A| = k$, then $|P(A)| = 2^k$

Let set A have $k+1$ elements $\quad |A| = k+1$

$A = \{a_1, a_2, a_3, \ldots, a_k, a_{k+1}\}$

Let set $A' = A - \{a_1\}$ $\quad |A'| = k$

$= \{a_2, a_3, \ldots, a_k, a_{k+1}\}$ $\quad |P(A')| = 2^k$ by IH

Subsets of A

- Some contain a_1
- Some do not contain a_1

Count each, add together to get $|P(A)|$

Subsets of A that contain a_1

$\boxed{2^k}$

$\{a_1\}$, $\{a_2, a_1\}$, $\{a_2, a_3, a_1\}$

Subsets of A without a_1
= all subsets of A'

$\boxed{2^k}$

$\{\}$, $\{a_2\}$, $\{a_2, a_3\}$ ...

total subsets of A $= |P(A)| = 2^k + 2^k = 2 \cdot 2^k = 2^1 \cdot 2^k = \boxed{2^{k+1}}$ ✓

Subsets without a_1

$\{\}$, $\{a_2\}$, $\{a_3\}$, $\{a_2, a_3\}$ ...

Subsets with a_1

$\{a_1\}$ $\{a_1, a_2\}$ $\{a_1, a_3\}$ $\{a_1, a_2, a_3\}$

$A = \{2, 3, 5, 7\}$ $A' = \{3, 5, 7\}$

Subsets of A without 2 $\{3\}$, $\{5\}$, $\{3, 5\}$, $\{3, 5, 7\}$, ...

Subsets of A with 2

$\{2\}$, $\{2, 3\}$, $\{2, 3, 5\}$, $\{2, 7\}$