CS5002
Feb 28th - Week 8 :)

Admin

- HW5 due 3/18 6pm
  (no discrete math over break :))
- ICA 6 due 2/29 6pm
- no class next week!
- no OH over break
- Late deadline HW5-7 $\boxed{4/17}$

Agenda

1. Pigeonhole Principle
2. Basic Probabilities

ICA 6

3. Expected Value
4. Conditional Probabilities
5. Bayes theorem

0. Context

Learned about sets, counting
using them to...
↳ make guarantees
↳ compute what's likely to happen

ceiling $\lceil x \rceil$ next ↑ integer

$\lceil 4.5 \rceil = 5$ $\quad \lceil 4.001 \rceil = 5$

$\lceil 4.9 \rceil = 5$ $\quad \lceil 5 \rceil = 5$

$\lceil 4 \rceil = 4$

Summation $\sum$

$$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6$$

Probabilities 0-1

0.25 = 25%
(0-1) (0-100)

joinpd.com
gnw kux

$\lceil N/k \rceil$

Pigeonhole ~ "guarantee" ~ what's N? what's k? what are we solving for?

2. Probability

↳ sets / counting → valid outcomes
probability → likelihood of the valid outcomes

- sum rule
- product rule
- subtraction rule
- set cardinality

probability measures the likelihood of **events**

Experiment - repeatable procedure with well-defined **set** of outcomes

Sample space - set of all possible outcomes of an experiment

$$S = \{s_1, s_2, \ldots, s_n\}$$

any subset of sample space is an **event**

$$E \subseteq S \qquad E_1 = \{s_1\} \qquad E_2 = \{s_2, s_3, s_4\} \qquad E_3 = \{s_1, s_6\}$$

$$\boxed{Pr(E) = \sum_{s \in E} p(s) = \frac{|E|}{|S|}}$$

ex Dice 6-sided

experiment - roll one die S = all outcomes $= \{1, 2, 3, 4, 5, 6\}$

- $E = \{3\}$ $Pr(E) = \frac{|E|}{|S|} = \frac{1}{6} = .166 = 16.6\%$
- $E = \{3, 5\}$ $Pr(E) = \frac{|E|}{|S|} = \frac{2}{6} = .333 = 33.8\%$

↳ sum rule! $Pr(E) = \sum_{s \in E} Pr(s) = Pr(3) + Pr(5) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = .333$

experiment - roll two dice - Die #1, then die #2

- $Pr(3 \text{ and then } 4)$? $1/6 \cdot 1/6 = 1/36 = .027 = 2.7\%$

$$P(3) \cdot P(4)$$

experiment - roll two dice

↳ sample space $\boxed{\text{- valid outcomes - } (5,4), (2,6), (5,5), \ldots}$

$\{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\} = \{(1,1), (1,2), (1,3) \ldots (6,6)\}$

$|6| \qquad |6| \qquad = 36$

$|S| = 36$ Pr (rolling 3, 4 in either order)? $|E| = 2$

$Pr(3 \text{ or } 4) = 2/36 = 1/18$

↳ sum rule! $Pr(3 \text{ or } 4) = Pr(3,4) + Pr(4,3) = 1/36 + 1/36 = 1/18$

experiment: roll 2 dice

Pr (sum is 4) $E = \{(1,3), (2,2), (3,1)\}$ $3/36 = .083$

+

Pr(sum is not 4) $|E| = 36 - 3 = 33$ $33/36 = .917$

$\overline{\quad 1 \quad}$

7:26

ICA#6

For this ICA, we'll consider the game of craps. You roll two six-sided dice. If the sum of the two outcomes is 7 or 11, you win! If the sum of the two outcomes is 2, 3, or 12, you lose. :(Otherwise, it's a draw.

The first time Laney played this game in a casino she got really annoyed because apparently there are 3 ways to lose and only two ways to win. So unfair! Right??

Problem #1

How many times would you have to roll two dice to guarantee you saw the same sum at least twice?

↳ pigeonhole

$\left\lceil \frac{N}{k} \right\rceil = 2 \quad k = 11$

k (boxes) - possible sums

N (objects) - dice rolls

$\left\lceil \frac{N}{11} \right\rceil = 2$ $N = 12$

Problem #2

What is the probability of winning a game of craps?

$|S| = 36$ $E = \{(6,1),(3,4),(5,2),(2,5),(4,3),(1,6),(5,6),(6,5)\}$

$Pr(E) = \frac{|E|}{|S|} = 8/36 = 2/9$

3. Expected Value

- experiment
- do experiment over + over, on average... what happens?

We answer with random variable

X value is associated with outcome of experiment

$E[X]$ is the expected value of X when we repeat the experiment ∞ and take the average

X needs to be numeric

$\boxed{E[X] \text{ is prob of every outcome times its value}}$

Formulas

$E[X] = \sum Pr(S_i) \cdot X_i$

$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$

Jeopardy winner 2023, computer scientist
70% chance of knowing the answer
30% chance of being wrong

"real life" example

Bet \$15,000 every bettable question

$(.70)(15000) + (.30)(-15000) = \$6{,}000$

$Pr(S_i) \cdot X_i$ $Pr(S_i) \cdot X_i$

experiment: rolling one die

X value associated with outcome

$$E[X] = \sum Pr(S_i) \cdot X_i$$

$Pr(X=1) = 1/6$
$Pr(X=2) = 1/6$
$Pr(X=3) = 1/6$
$Pr(X=4) = 1/6$
$Pr(X=5) = 1/6$
$Pr(X=6) = 1/6$

$$1/6 \cdot 1 + 1/6 \cdot 2 + 1/6 \cdot 3 + 1/6 \cdot 4 + 1/6 \cdot 5 + 1/6 \cdot 6$$

$$\boxed{= 3.5}$$

ex roll two dice $|S| = 36$

X = random variable associated with # of #s > 2

$Pr(X=0)$ $\{(1,2), (2,2), (1,1), (2,1)\} = 4/36$

$Pr(X=1)$ $\{(1,x), (x,1), (x,2), (2,x)\} = 16/36$

$Pr(X=2)$ $\{(3,3), (3,4), (3,5) \ldots \} = 16/36$

$$E[X] = \sum Pr(S_i) \cdot X_i = 4/36 \cdot 0 + 16/36 \cdot 1 + 16/36 \cdot 2 = 1.33$$

4. Conditional Probability

So far... independent probabilities

ex roll a die, $Pr(3,3,3)$

$$1/6 \cdot 1/6 \cdot 1/6 = 1/6^3$$

conditional probability: some other event has occurred, impacting the probability of original event

E, F are events

- independently $Pr(E)$
- conditionally $Pr(E|F)$ pr. of E given F

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$$

ex Bit string of length 4
all outcomes equally likely

experiment: generate the bit string
$|S| = 2^4 = 16$

$E = \{x \mid x \text{ has at least 2 consecutive zeroes}\}$

↳ cases!

exactly 4 — 1 of these

OR

exactly 3 — 0 0 0 1, 1 0 0 0 — 2 of these

OR

exactly 2 — 0 0 1 x, 1 0 0 1, x 1 0 0 — 5 of these

$|E| = 1 + 2 + 5 = 8$

$Pr(E) = 8/16 = .5$

↳ add more info
some event F has occurred! } First bit is a zero (F)

$Pr(E) = 1/2 \quad Pr(F) = 1/2$

0 ? ? ?

$Pr(E|F) = \dfrac{Pr(E \cap F)}{Pr(F)}$

↳ $E \cap F = \{0000, 0001, 001x, 0100\}$

$|E \cap F| = 5 \quad Pr(E \cap F) = 5/16$

$Pr(E|F) = \dfrac{5/16}{8/16} = \dfrac{5}{16} \cdot \dfrac{16}{8} = \dfrac{5}{8} = .625 = 62.5\%$

What if ...? F = starts with 1

$Pr(E) = 1/2 \quad Pr(F) = 1/2$

$E \cap F = \{1000, 1001, 1100\}$

$|E \cap F| = 3 \quad Pr(E \cap F) = 3/16$

$Pr(E|F) = \dfrac{3/16}{8/16} = \dfrac{3}{16} \cdot \dfrac{16}{8} = 3/8 = .375$

5. Bayes Theorem

$Pr(\text{have a disease} \mid \text{diagnosed})$

↳ more information
conditional probability

Putting together what we know of ind., conditional probabilities

Insight #1

$Pr(E \cap F)$ in conditional prob formula $\quad Pr(E|F) = \dfrac{Pr(E \cap F)}{Pr(F)}$

maybe we don't know $E \cap F$ in real world

$$Pr(E \cap F) = Pr(E|F) \cdot Pr(F)$$
$$= Pr(F|E) \cdot Pr(E)$$

$$Pr(F|E) = \frac{Pr(F \cap E)}{Pr(E)}$$

Insight #2

$Pr(F)$ in conditional prob formula

maybe we don't know $Pr(F)$ in real world

- but, we know F could happen, or not happen
- F could happen with or without E

$$Pr(F) = Pr(F \cap E) + Pr(F \cap \neg E)$$
$$= Pr(F|E) \cdot Pr(E) + Pr(F|\neg E) \cdot Pr(\neg E)$$

Bayes

$$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)} = \frac{Pr(F|E) \cdot Pr(E)}{Pr(F)}$$

$$Pr(F) = Pr(F|E) \cdot Pr(E) + Pr(F|\neg E) \cdot Pr(\neg E)$$

ex) Tottenham, Sonny the star player

we know:

- Sonny plays in 20/38 games
- we win 75% of games when Sonny plays
- we win 40% of games when he doesn't play

we want to know...

Pr(win)? ⟶ $Pr(W) = Pr(W|S) \cdot Pr(S) + Pr(W|\neg S) \cdot Pr(\neg S)$

$(.75)(20/38) + (.40)(18/38)$

$= .58$

Pr(Sonny | won)?

$P(S|W) = \frac{Pr(S \cap W)}{Pr(W)}$ ⟸ $\cap$

$\frac{Pr(W|S) \cdot Pr(S)}{Pr(W)}$ ⟸ $\cup$

$= \frac{(.75)(20/38)}{.58} = \boxed{.6805}$

$\frac{Pr(F|E) \cdot Pr(E)}{Pr(F)}$

Bonus EV example

X needs to be numeric

experiment - flipping a coin - Heads or Tails

$Pr(H) = Pr(T) = 1/2$

X associated with the outcome (needs to be numeric)

⟶ experiment - flip a coin 3 times

X value associated with # of tails

E[X] ~ avg # of tails in 3 flips

outcomes = {HHH, TTT, TTH, THT, HTT, HHT, HTH, THH}

$Pr(X=0) = 1/8$

$Pr(X=1) = 3/8$

$Pr(X=2) = 3/8$

$Pr(X=3) = 1/8$

$E[X] = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.5$

Indicator RV ~ Focus on one coin flip

$X = \#$ of tails $\qquad E[X] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E[X]$ for 3 flips $= E[X_1] + E[X_2] + E[X_3]$

$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1.5$

$E[X]$ for 10 flips $= \frac{1}{2} \cdot 10 = 5$

$E[X]$ for 100 flips $= \frac{1}{2} \cdot 100 = 50$