CS5002

Feb 7th - Week 5

Admin

- HW3 was due 6pm
- HW4 due 2/14 6pm

Exam #1 - 2/21

- in class, usual time
- no lecture
- scanned afterwards
- content up through HW4
- on paper
- 6-9:20pm (but assigned shorter)
- practice problems review next week!
- cheat sheet 8.5x11, one side
- no other materials

Agenda

1. Inclusion/exclusion principle (IEP)
2. Product rule
3. Sum rule

ICA #5

4. Combinations
5. Permutations

0. Review

- one thing we learned
- one Q we still have

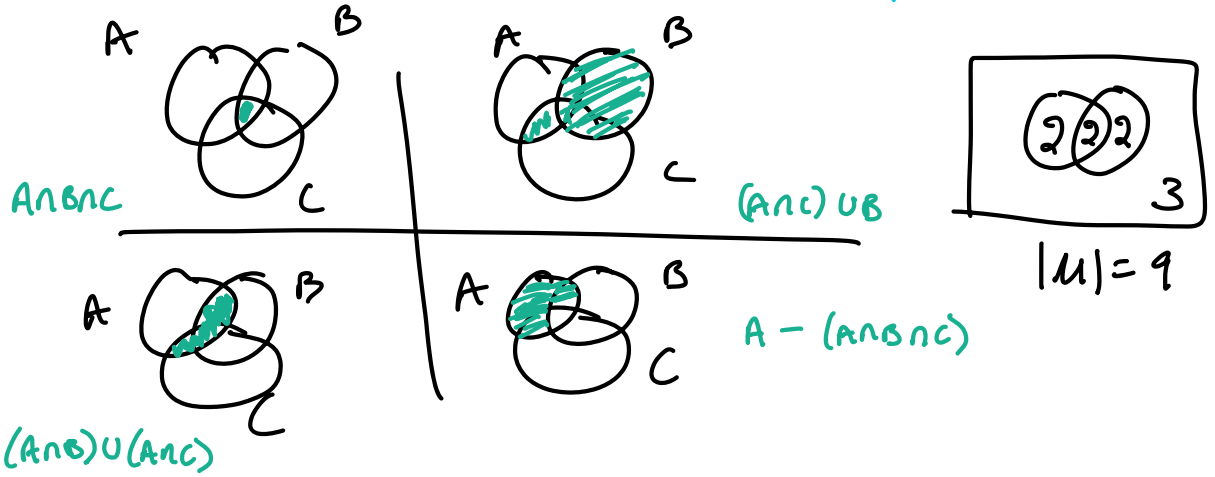
set equality

cardinality $|S|$

set operators $\cup, \cap, -$

Powerset

Set builder notation $S = \{x \mid x \text{ is odd} \wedge x \text{ is positive}\}$



1. Inclusion/Exclusion Principle

- want to be able to count cardinality as it appears in a venn diagram

Deck of cards:
- 52 cards
- ½ red, ½ black
- value (2, 3, 4, ..., 10, J, Q, K, A)
- suit (clubs, diamonds, hearts, spades)

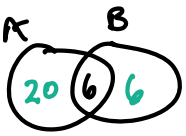
M = entire deck
$A = \{x \mid x \text{ is red}\}$ $B = \{x \mid x \text{ is a face card}\}$

$|A| = 26$ $|B| = 12$

$|A \cap B| = 6$
$|A - B| = 20$
$|A \cap B| = 20$
$|B - A| = 6$
$|S| = 40$

$|A \cup B| = |A| + |B|$? 1,1 ∧ XXX
$= 26 + 12$
$= 38$?

red face cards get counted twice!



$|A \cup B| = 32$

In general... $|A \cup B| = |A| + |B| - |A \cap B|$ IEP

How many ways...?

2. Product Rule

- tool to answer "how many ways--?"
  - operations does my prog need?
    hack my password?
    interact addresses?
    compare feature sets?

tasks
- n ways to do task 1
- m ways to do task 2

↳ there are $n \cdot m$ ways to do task 1 (and) task 2

(ex) $L = \{ww, rodeo\}$ $T = \{fg, archer\}$

$L \times T = \{(l, t) \mid l \in L \wedge t \in T\}$

$|L \times T| = |L| \cdot |T|$

task 1: pick an L show
task 2: pick a T show

$|L|$ ways to do task 1
$|T|$ ways to do task 2

How many ways to spend an evening at Stonycrave?

(ex) How many subsets are there of a given set? ≡ $|P(S)|$?

Set S
$P(S) = \{A \mid A \subseteq S\}$

from last week: $|P(S)| = 2^{|S|}$
↳ Because: product rule

$S = \{a, b, c\}$ — a can be in subset, or not
b can be in subset, or not
c can be in subset, or not

$\{a\}, \{b, c\}$
$\{a, b, c\}$

0 - not in
1 - in

task 1: decide if a in subset
task 2: decide if b in subset
task 3: decide if c in subset

↳ $2 \cdot 2 \cdot 2 = 2^3 = 2^{|S|}$

a	b	c	subsets
0	0	0	$\{\}$
0	0	1	$\{c\}$
0	1	0	$\{b\}$
1	0	0	$\{a\}$
1	0	1	$\{a, c\}$
.	.	.	. . .

(ex) deck of cards, with replacement

How many ways are there to...

- pick two cards?
- pick a Queen and then a King?
- pick two queens?

	task 1		task 2	
	52	·	52	= 2704
	4	·	4	= 16
	4	·	4	= 16

valid outcome: QH, KS ✓
KS, QH? XX

↳ valid outcome: QD, QH ✓
QH, QD ✓

but are they the same?
Does or (c) count as 1 or 2 things?

$A = \{QD, QH, QS, QC\}$ $B = \{QD, QH, QS, QC\}$

$A \times B = \{(QD, QH), (QH, QD) \ldots\}$
A B A B

$|A \times B| = |A| \cdot |B|$
$= 4 \cdot 4$
$= 16$

order matters!

(ex) Bit string: sequence of 0s and 1s

How many bit strings of length 7?

_ _ _ _ _ _ _
0/1 0/1 0/1 0/1 0/1 0/1 0/1

task: choose a bit
2 ways
7 tasks 2^7

How many bit strings of length 7 start with 0?

↳ $1 \cdot 2^6 = 2^6$
↳ always a zero

valid outcomes -
0111111 ✓ 011 X
1111111 X 0000000 ✓

3. Sum Rule n ways to do task 1, m ways to do task 2 } there are n+m ways to do task 1 (or) task 2

How many ways ... ?

- Select a Queen or a King? task 1: 4, task 2: 4 — 4+4=8
- Select a Queen and King in either order? } Select a KQ (or) QK?

  (4·4) + (4·4) = 32

Valid outcomes: QD, KH ✓
KH, QD ✓

(ex) Bit strings:

How many bit strings of length 1-5?

= length 1 or length 2 or length 3 or length 4 or length 5?

$2^1 + 2^2 + 2^3 + 2^4 + 2^5$

$= 62$

(ex) alpha strings (A-Z)

How many alpha strings of length 6? $26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^6$

How many alpha strings of length 6 have only vowels? $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^6$

valid outcomes = aeiou2 ✓
ciciic ✓
bciicii ×

How many alpha strings of length 6 have at least one consonant?

"at least" in cases
1 conson. 3 conson
2 conson. 4 conson
5 cons 6 conson

We could compute each case indv.

Sometimes better! → total outcomes − invalid outcomes

↳ total outcomes: all alpha strings of length 6: 26^6
invalid outcomes: no consonants, all vowels: 5^6

$26^6 - 5^6 = 308,900,151$ 7:51

LCA#5

Problem #1

If there are *n* Khoury faculty members in Boston, how many ways are there to form a faculty committee of any size?

n=3? a, b, c; 0/1 0/1 0/1 every fac could be on cmte, or not → 2^n

Problem #2

How many ways to sit Laney's dog Grizz, and 5 of his dog friends, on Laney's couch so she can take a picture?

$\_ \cdot \_ \cdot \_ \cdot \_ \cdot \_ \cdot \_$

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 6! = 720$

task: choose a dog to sit at the position

14. Permutation

- How many ways are there ...?
- Using the product rule (choose a dog, can't choose them again)
- but, no repetition

ex: bit strings valid outcomes: 0000111

k tasks
n ways for each task

Product rule: n^k → w/ repetition

now... product rule w/ no repetition (permutation)
- arrange the elements of a set
- n elements in set, arrange k of them

ex) 5 cards {10, 5, A, 7, 3} n = 5

arrange all 5 ... 5 tasks $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$

5 ways to do task 1
4 ways to do task 2
3 ways to do task 3
2 ways to do task 4
1 way to do task 5

valid outcome =

diff outcomes: 2, 7, 3, 5, 10 / 7, 3, 2, 10, 5

ex) 5 cards {10, 5, 2, 7, 3} n = 5, k = 3

arrange 3 cards ... 3 tasks $5 \cdot 4 \cdot 3 = 60$

Permutation
- n elements
- arrange k of them

$$P(n,k) = \frac{n!}{(n-k)!}$$

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ = arrange all 5 cards $(n!)$

$5 \cdot 4 \cdot 3$ = arrange 3 of 5 cards $(n!)$ / not want

$\frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} = 5 \cdot 4 \cdot 3$ cancel out invalid arrangements

ex) 51 people
select a firefighter, and a firestarter

we want: $51 \cdot 50$

$P(n,k)$ $n = 51$, $k = 2$

$$\frac{51!}{(51-2)!} = \frac{51!}{49!} = \frac{51 \cdot 50 \cdot 49 \cdot 48 \cdots 1}{49 \cdot 48 \cdots 1} = 51 \cdot 50$$

Permutation formula applies when: tasks, no repetition, order matters

5. Combinations — How many ways are there ... ?

- n elements in a set
- choose k of them
- two outcomes are the same if they contain the same elements
- order doesn't matter
- no repetition

$S = \{L, C, PB, T\}$ of cookies
I want to eat 2 of them!
$n = 4 \quad k = 2$

Valid outcomes:

L, C	PB, T
PB, L	C, T
L, T	C, PB

$\{T, PB\}$? ✓

Permutation → Combination

L, C	C, T
C, L	T, C
L, PB	C, PB
PB, L	PB, C
L, T	PB, T
T, L	T, PB

- select or two cookies — 2
- perm: arrange the selection — 2!

How do we count a combination?
- start with permutation
- remove everything double counted

Start with: n elements
Choose k of them
permutation arranges k!

$$C(n,k) = \frac{n!}{k!(n-k)!}$$

↳ choose 2 of 4 cookies
$n = 4$
$k = 2$

$$C(4,2) = \binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{(2 \cdot 1)(2 \cdot 1)} = \frac{4 \cdot 3}{2} = \boxed{6}$$

!! ☺ 🎉

Apply combo formula when: order doesn't matter, no rep

(ex) How many bit strings of length 7 have exactly one zero? } How many ways to choose the position of the zero?

$n = 7$ (positions with the zero)
$k = 1$ (position to choose)

$$C(7,1) = \frac{7!}{1!(7-1)!} = \frac{7!}{6!} = 7$$

(ex) How many bit strings of length 125 have exactly 18 zeroes?

$C(125, 18)$

(ex) How many bit strings of length 4 have one zero?

Sample: 0111 or 1011 or 1101 or 1110

(ex) How many bit strings of length 4 have 2 zeroes?

Sample:
0011
0101
0110

(ex) How many bit strings of length 10 have at most 3 zeroes?

cases!
- exactly 3 zeroes: $C(10,3)$ OR
- exactly 2 zeroes: $C(10,2)$ OR
- exactly 1 zero: $C(10,1)$ OR
- exactly 0 zeroes: 1

$C(10,3) + C(10,2) + C(10,1) + 1 = 176$

order matters, no rep: $P(n,k)$	order matters, rep ok: n^k
order doesn't matter, no rep: $C(n,k)$	order doesn't matter, rep ok: (next week)

Any counting question...
- what are valid outcomes?
- is rep ok?
- does order matter?