

Admin

- HW7 was due 6pm - no new HW :)
- ICA #9 today - last ICA :)
- Exam #2 \cup 4110 during class
- 4117 late deadline HW5-7
- 4124 make-up exam question
- 4117 lecture - runtime algorithms

Agenda

1. Sequences
2. Types of Sequences
3. Deriving a formula
< ICA #9 >
4. Summations
5. Practice Problems
6. TRACE

1. Sequences

Discrete Structure: collection of objects

set - unordered collection of distinct objects

graph - (V, E) where V is a non-empty set of vertices
 E is a set of edges

\hookrightarrow also: tree!

\star Sequence - ordered list of objects which may be infinite

\hookrightarrow ability to identify, characterize patterns
 study repeated processes

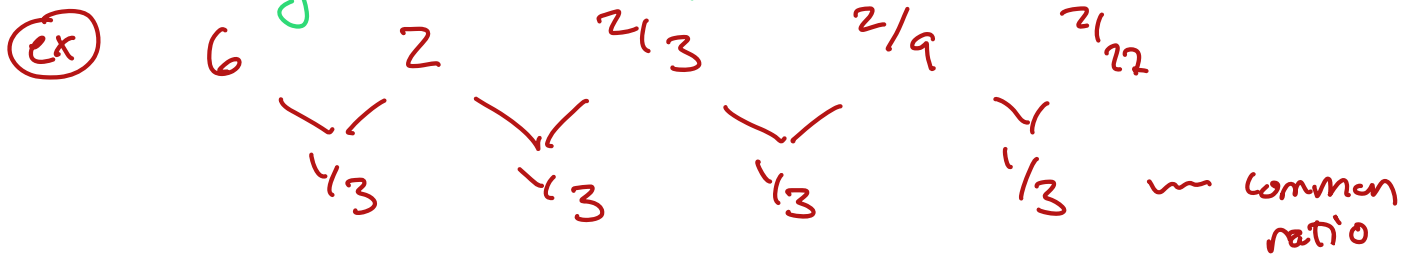
notation:-

$\{a_n\}$ represents a sequence

a_k \sim arbitrary element of sequence, at position k

finite $\{a_1, a_2, \dots, a_n\}$ } list elements of a sequence
 infinite $\{a_1, a_2, \dots\}$ } ordered!

Geometric: multiply same value to previous term



Approach, given first few terms...

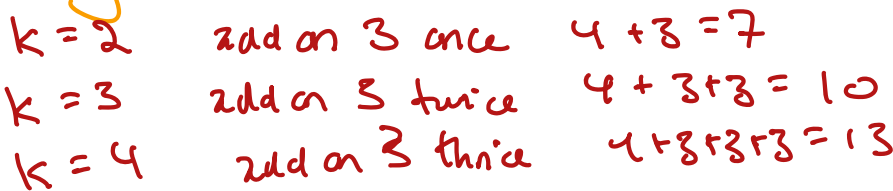
1. How much do we add every time? Always the same = arithmetic
2. How much do we multiply every time? Always the same = geometric

Arithmetic Formula for a_k

• goal: compute value at a_k without having to know all previous terms



How many times do we add on 3, relative to k ? — $k-1$ times!



a_k needs: common difference (d), initialize value (a_1)

$d=3 \quad a_1=4$

$a_k = a_1 + (k-1)d$ ★ any arithmetic sequence

Confirm - formula works?

$a_1 = a_1 + 0 \cdot 3 = 4$

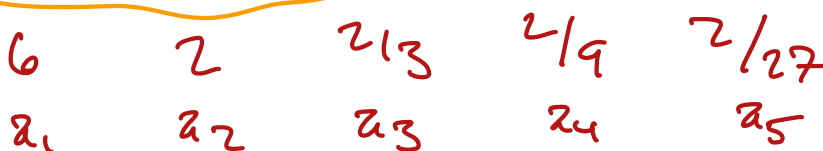
$a_2 = a_1 + (1)(3) = 4 + 3 = 7$

$a_3 = a_1 + (2)(3) = 4 + 6 = 10$

$a_4 = a_1 + (3)(3) = 4 + 9 = 13$



geometric formula for a_k



multiply by common ratio to get next term

Common ratio = $\frac{1}{3}$ starting point is 6

How many times do we multiply $\frac{1}{3}$ to get to term a_k ?

$k=2$ multiply by $\frac{1}{3}$ once $6 \cdot \frac{1}{3} = 2$

$k=3$ multiply by $\frac{1}{3}$ twice $6 \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{3}$

$k=4$ multiply by $\frac{1}{3}$ thrice $6 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$

$k-1$

need for geometric formula: common ratio (r)
initial value (a_1)

$a_k = a_1 \cdot r^{(k-1)}$ any geometric sequence!

confirm the formula...

$a_1 = a_1 \cdot r^0 = 6 \cdot (\frac{1}{3})^0 = 6$

$a_2 = a_1 \cdot r^1 = 6 \cdot (\frac{1}{3})^1 = \frac{6}{3} = 2$

$a_3 = a_1 \cdot r^2 = 6 \cdot (\frac{1}{3})^2 = 6 \cdot \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

CA #9

Problem #1

Find the first five terms a_1, a_2, a_3, a_4, a_5 of the sequence given by the following formula:

$a_k = \frac{k-1}{k}$

- $a_1 = 0$
- $a_2 = \frac{1}{2}$
- $a_3 = \frac{2}{3}$
- $a_4 = \frac{3}{4}$
- $a_5 = \frac{4}{5}$

not arithmetic
not geometric

Problem #2

For each sequence below, determine whether it is arithmetic. If it is arithmetic, find the common difference, the initial value, and give the formula to calculate a_k .

-23, -20, -17, -14, ... arithmetic! common diff = 3 initial value = -23

$$a_k = -23 + (k-1)(3)$$

-24, 4, 34, 64, ...

not arithmetic

-30, -28, -26, -24, ... arithmetic! diff = 2 init value = -30

$$a_k = -30 + (k-1)(2)$$

4. Summations

Sequence things we care about

- what is the next value?
- what is value at a_k ?
- what are the first few terms?
- what is sum of first n terms?

→ partial sum

- add together $a_1 + a_2 + \dots + a_n$
- where $n \in \mathbb{Z}^+$
- if we know sequence terms, can add by hand
- better to have a formula!

(ex) 2 4 8 16 32

Add together...

- first 1 term? 2
- first 2 terms? $2 + 4 = 6$
- first 3 terms? $2 + 4 + 8 = 14$
- first 4 terms? $2 + 4 + 8 + 16 = 30$

$$\sum_{i=1}^n a_i$$

$$\sum_{i=1}^1 a_i = a_1 = 2$$

$$\sum_{i=1}^2 a_i = a_1 + a_2$$

$$\sum_{i=1}^3 a_i = a_1 + a_2 + a_3$$

$$\sum_{i=1}^4 a_i = a_1 + a_2 + a_3 + a_4$$

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n \quad \text{by definition!}$$

In this case, we find a formula! Sum of first n terms is always 2 less than power of 2 — formula $2^{(n+1)} - 2$

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n = 2^{(n+1)} - 2$$

definition, formula for sum of first n terms

(ex) sequence $\{a_n\}$ given by 1, 2, 3, 4, 5... (arithmetic!)

Start with: summation by definition

$$\sum_{i=1}^1 a_i = a_1 = 1$$

$$\sum_{i=1}^2 a_i = a_1 + a_2 = 1 + 2 = 3$$

$$\sum_{i=1}^3 a_i = 1 + 2 + 3 = 6$$

Add first n terms...

$$n=1 \quad \text{sum}=1$$

$$n=2 \quad \text{sum}=3$$

$$n=3 \quad \text{sum}=6$$

$$n=4 \quad \text{sum}=10$$

$$n=5 \quad \text{sum}=15$$

$$n=6 \quad \text{sum}=21$$

sequence: 1, 2, 3, 4... $a_k = k$

sequence

of summations: 1, 3, 6, 10, 15, 21

$$a_k = \frac{(k)(k+1)}{2}$$

sum of first n terms: $\frac{(n)(n+1)}{2}$

$$\sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n = \frac{(n)(n+1)}{2}$$

8:00

goal: prove definition + formula are the same!

n

defn

formula

1

1

$$\frac{(1)(2)}{2} = 1$$

$$2 \quad 1+2=3 \quad \frac{(2)(3)}{2} = 3$$

$$3 \quad 1+2+3=6 \quad \frac{(3)(4)}{2} = 6$$

⋮

(we could show one-by-one for 2, 3, 4, ... values of n ... induction is a shortcut!)

Predicate $P(x) \quad \sum_{i=1}^x i = \frac{(x)(x+1)}{2}$

Logic st. $\forall x \in \mathbb{Z}^+ P(x)$

Base case $P(1) \quad \sum_{i=1}^1 i = \frac{(1)(2)}{2} \quad \checkmark \quad (\text{above!})$

Inductive step $P(k) \Rightarrow P(k+1)$

IH $P(k)$ is true, $\sum_{i=1}^k i = \frac{(k)(k+1)}{2}$

Goal $P(k+1)$
 $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=1}^{k+1} i = 1+2+3+\dots+k+k+1$$

// by def

$$= \frac{(k)(k+1)}{2} + (k+1)$$

// by IH

$$= \frac{(k)(k+1)}{2} + \frac{2(k+1)}{2}$$

// multiply 2nd term by $\frac{2}{2}$

$$= \frac{(k)(k+1) + 2(k+1)}{2}$$

// add numerators

$$= \frac{(k^2+k) + 2k+2}{2}$$

// expansion

$$= \frac{k^2 + 3k + 2}{2}$$

// adding like terms

$$= \frac{(k+1)(k+2)}{2} \quad \checkmark$$

// binomial

$$(k+1)(k+2)$$

$$FOIL = k^2 + 2k + k + 2$$

maybe

13

11

9

6

Practice #6

Practice #6 -- Probability

Prof. Lucia Nunez (she/her) has a vast collection of plushies. She wants to bring one plushie for her trip to the Berkshires at the end of the semester. She owns two sets of plushies, one set is in Boston and one is in New York.

Boston Plushies (12 total):

- 4 blue ones
- 2 pink ones
- 4 white ones
- 2 brown ones

New York Plushies (13 total):

- 4 white ones
- 3 gray ones
- 2 blue ones
- 4 pink ones

Lucia will choose from the Boston set 7/11 times, and from the New York set 4/11 times. One she decides on a set, she picks a plushie from the set at random.

A What is the probability that Lucia will choose from the new york set and choose a pink plushie?

$$Pr(NY) \cdot Pr(\text{pink in NY})$$

$$\left(\frac{4}{11}\right) \cdot \left(\frac{4}{13}\right)$$

B What is the probability that she will choose a blue plush?

$$NY \text{ blue} + Boston \text{ blue}$$

$$\left(\frac{4}{11}\right)\left(\frac{2}{13}\right) + \left(\frac{7}{11}\right)\left(\frac{4}{12}\right)$$

C Lucia went on her trip with a blue plushie. What is the probability she chose from the NY set, given that her plushie was blue?

$$Pr(NY | \text{blue})$$

$$Pr(\text{blue}) = \left(\frac{4}{11}\right)\left(\frac{2}{13}\right) + \left(\frac{7}{11}\right)\left(\frac{4}{12}\right)$$

$$Pr(NY) = \frac{4}{11}$$

$$Pr(\text{blue} | NY) = \frac{2}{13}$$

$$Pr(NY | \text{blue}) = \frac{Pr(NY) \cdot Pr(\text{blue} | NY)}{Pr(\text{blue})} = \frac{\frac{4}{11} \cdot \frac{2}{13}}{\frac{4}{11} \cdot \frac{2}{13} + \frac{7}{11} \cdot \frac{4}{12}} = .208$$

Practice #9 -- Probability

How many times would you expect to flip a coin in order to see two consecutive heads?

Expected value of a random variable:

$$\text{flip a coin 3 times} \quad Y_2 = Y_2 + Y_2 = 1.5 \text{ heads}$$

Flipping the problem to answer how many times we'd flip the coin
- want 2 consecutive heads

H/T H/T H/T H/T ...
2 heads in a row,
2nd stop

valid outcomes

2 cons. heads



what we wanted!

2 flips
 $Pr(HH) = 1/4$

1 cons. head



2 + E flips

$Pr(HT) = 1/4$

0 cons. heads



1 + E flips

$Pr(T) = 1/2$

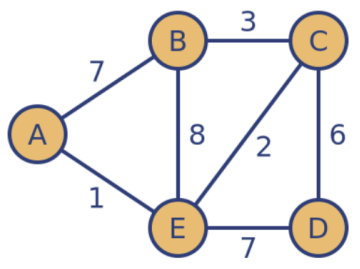
Expected # of flips = $(2)(1/4) + (2+E)(1/4) + (1+E)(1/2)$

E = expected # flips $E = 1/2 + 1/2 + E/4 + 1/2 + E/2$

$E = 6$

Practice #11 - Graphs

This question is concerned with the following graph.



Use Dijkstra's algorithm to find the total weight of the shortest path from A to C. To show your work, fill in the table below (feel free to scratch out/replace values as you update them, or rewrite the table, as long as we can follow how you got there).

	A	B	C	D	E
Predecessor	--	A, C	E	E	A
Path Weight	0	7, 6	3	8	1

total weight A, E, C = 3

start: A

step 2: E

step 3: C

step 4: B
(no change)

step 5: D
(no change)

Practice #12 -- induction

When we cover the growth of functions in the next few weeks, we'll regularly use the assumption that exponential functions grow slower than factorial functions, e.g., that $3^n < n!$. Prove by induction that the statement is true for all integers $n \geq 7$.

Predicate $S(n)$ states that $3^n < n!$

Logic Statement $\forall n \in \mathbb{Z}^+ n \geq 7 \Rightarrow S(n)$

Base Case _____ $S(7)$ $3^7 = 2187$ $7! = 5040$ $3^7 < 7!$ ✓

Inductive Step $S(k) \Rightarrow S(k + 1)$ for arbitrary $k \geq 7$

IH $S(k)$ $3^k < k!$

goal: $3^{k+1} < (k+1)!$

$$3^{k+1} = 3^k \cdot 3 < k! \cdot 3$$

// by IH

$$\left\{ \begin{array}{l} < k! \cdot 3 \cdot 3 \\ < k! \cdot k! \end{array} \right.$$

anything, as long as ineq. is ok

$$< (k!)(k+1)$$

// because $k \geq 7$

$$= (k+1)!$$

// def of factorial

□ done!