6 Jacobians

6.1 Linearized Kinematics

In previous chapters we have seen how kinematics relates the joint angles to the position and orientation of the robot's end-effector. This means that, for a serial robot, we may think of the forward kinematics as a mapping from joint space to the space of rigid body motions. The image of this mapping is the work space of the robot. In general, the work space will be only a subspace of the space of all rigid body motions; it consists of all positions and orientations reachable by the robot's end-effector. As we have already mentioned, we can only put local co-ordinates or parameterisations on the space of rigid body motions†.

We can also consider mappings associated with particular points; note that the image of such a map is sometimes called the work space of the robot: it is the space of point reachable by some point on the end-effector. Consider, for example the wrist centre of a Puma robot:-

$$\mathbf{K} : (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \longrightarrow (x'_c, y'_c, z'_c)$$

The map is given explicitly in terms of the $\mathbf{A}$-matrices:-

$$\begin{pmatrix} x'_c \\ y'_c \\ z'_c \\ 1 \end{pmatrix} = \mathbf{A}_1(\theta_1)\mathbf{A}_2(\theta_2)\mathbf{A}_3(\theta_3)\mathbf{A}_4(\theta_4)\mathbf{A}_5(\theta_5)\mathbf{A}_6(\theta_6) \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix}$$

Here, (x_c, y_c, z_c) are the home co-ordinates of the wrist centre. In other words we have three functions:-

$$\begin{aligned} x'_c &= k_1(\theta_1, \dots, \theta_6) \\ y'_c &= k_2(\theta_1, \dots, \theta_6) \\ z'_c &= k_3(\theta_1, \dots, \theta_6) \end{aligned}$$

† Some authors like to regard the forward kinematics as a co-ordinate transformation; but this is not possible since the spaces concerned are topologically different.

As we have seen previously, these are highly non-linear functions of the joint angles. However, if we are only interested in the neighbourhood of some point, it is possible to linearize the map. That is, we find a linear approximation to the original map. So if we make small changes in the joint angles we get:-

$$\begin{aligned}
\delta x &= \frac{\partial k_1}{\partial \theta_1}\delta\theta_1 + \frac{\partial k_1}{\partial \theta_2}\delta\theta_2 + \cdots + \frac{\partial k_1}{\partial \theta_6}\delta\theta_6 \\
\delta y &= \frac{\partial k_2}{\partial \theta_1}\delta\theta_1 + \frac{\partial k_2}{\partial \theta_2}\delta\theta_2 + \cdots + \frac{\partial k_2}{\partial \theta_6}\delta\theta_6 \\
\delta z &= \frac{\partial k_3}{\partial \theta_1}\delta\theta_1 + \frac{\partial k_3}{\partial \theta_2}\delta\theta_2 + \cdots + \frac{\partial k_3}{\partial \theta_6}\delta\theta_6
\end{aligned}$$

If we write $(\delta x_c', \delta y_c', \delta z_c')^T = \mathbf{\Delta x}$ and $(\delta\theta_1, \ldots, \delta\theta_6)^T = \mathbf{\Delta}\boldsymbol{\theta}$, then we can summarize the above equations as:-

$$\mathbf{\Delta x} = \mathbf{J}\,\mathbf{\Delta}\boldsymbol{\theta}$$

The matrix $\mathbf{J}$ is called the **jacobian** of the map; that is, the jacobian is the matrix of partial derivatives. In this case:-

$$\mathbf{J} = \begin{pmatrix} \frac{\partial k_1}{\partial \theta_1} & \frac{\partial k_1}{\partial \theta_2} & \cdots & \frac{\partial k_1}{\partial \theta_6} \\ \frac{\partial k_2}{\partial \theta_1} & \frac{\partial k_2}{\partial \theta_2} & \cdots & \frac{\partial k_2}{\partial \theta_6} \\ \frac{\partial k_3}{\partial \theta_1} & \frac{\partial k_3}{\partial \theta_2} & \cdots & \frac{\partial k_3}{\partial \theta_6} \end{pmatrix}$$

The jacobian matrix behaves very like the first derivative of a function of one variable. For a function of several variables we have a version of Taylor's theorem:-

$$\mathbf{x} + \mathbf{\Delta x} \approx \mathbf{k}(\boldsymbol{\theta}) + \mathbf{J}(\boldsymbol{\theta})\mathbf{\Delta}\boldsymbol{\theta}$$

For small variations about $\boldsymbol{\theta}$ the map is approximated by its value at $\boldsymbol{\theta}$ plus $\mathbf{J}(\boldsymbol{\theta})$ times the variation, $\mathbf{\Delta}\boldsymbol{\theta}$.

For an example we turn to the planar manipulator yet again, see fig. 6.1. The kinematic equations of the end point are given by:-

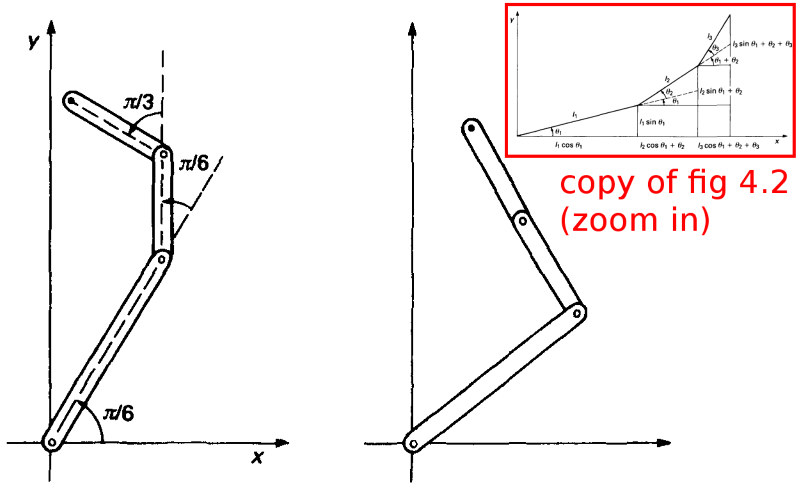
$$\begin{aligned}
x &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\
y &= l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3)
\end{aligned}$$

The jacobian of this is:-

$$\mathbf{J}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \end{pmatrix}$$

where:-

$$\begin{aligned}
\frac{\partial x}{\partial \theta_1} &= -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\
\frac{\partial x}{\partial \theta_2} &= l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\
\frac{\partial x}{\partial \theta_3} &= l_3\sin(\theta_1+\theta_2+\theta_3) \\
\frac{\partial y}{\partial \theta_1} &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3)
\end{aligned}$$



copy of fig 4.2
(zoom in)

Figure 6.1 The Final and Starting Positions for the Planar Manipulator Example

$$\begin{aligned}\frac{\partial y}{\partial \theta_2} &= l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3) \\ \frac{\partial y}{\partial \theta_3} &= l_3 \cos(\theta_1 + \theta_2 + \theta_3)\end{aligned}$$

6.2 Errors

One of the first uses we can make of the jacobian is to find the effect of errors in the joint angles. An error of $\boldsymbol{\Delta\theta}$ in the joint angles will produce a positional error of $\boldsymbol{\Delta}\mathbf{x} = \mathbf{J}\,\boldsymbol{\Delta\theta}$. Because the map is non-linear, the effect of errors will be different at different positions. Consider the planar manipulator in its home position; $\theta_1 = \theta_2 = \theta_3 = 0$.

$$\mathbf{J}(0,0,0) = \begin{pmatrix} 0 & 0 & 0 \\ l_1 + l_2 + l_3 & l_2 + l_3 & l_3 \end{pmatrix}$$

To first order, no joint error can produce an error in the x-direction. An error of $1/10$ of a radian in θ_2, $\boldsymbol{\Delta\theta} = (0, 0.1, 0)^T$, will give a y-error of:-

$$\delta y \approx 0.1(l_2 + l_3)$$

In a different position, say $\theta_1 = \theta_3 = 0, \theta_2 = \frac{\pi}{2}$, the jacobian is:-

$$\mathbf{J}(0, \frac{\pi}{2}, 0) = \begin{pmatrix} -l_2 - l_3 & -l_2 - l_3 & -l_3 \\ l_1 & 0 & 0 \end{pmatrix}$$

Now an error $\boldsymbol{\Delta\theta} = (0, 0.1, 0)^T$ will give a positional error of:-

$$\boldsymbol{\Delta}\mathbf{x} \approx \mathbf{J}(0, \frac{\pi}{2}, 0)\boldsymbol{\Delta\theta} = \begin{pmatrix} -\frac{1}{10}(l_2 + l_3) \\ 0 \end{pmatrix}$$

So there is no error in the y-direction and an error of $-\frac{1}{10}(l_2 + l_3)$ in the x-direction; to first order.

This tells us about singularities in the kinematics. In section 5.2 we defined a singularity as a point where the robot loses a degree-of-freedom. In fact at a singularity the robot loses an 'instantaneous' degree-of-freedom also. This means that, to first order, the robot's end-effector cannot move in one direction. The columns of the jacobian span the instantaneous directions the end-effector can move in. That is, the robot can only move in directions which are linear combinations of the columns of the jacobian. Thus a better definition of a singularity is as follows. A point ϕ in the joint space of a robot is a singular point if and only if the jacobian $\mathbf{J}(\phi)$ has less than maximal rank. That is, if there are linear dependencies among the columns of the jacobian.

In the example above, $\mathbf{J}(0, 0, 0)$ had a row of zeroes. So all the 2×2 submatrices would have zero determinant and thus the rank of the jacobian is one. Hence, the home position is singular. However, $\mathbf{J}(0, \frac{\pi}{2}, 0)$ has a submatrix with non-zero determinant, so the rank is two, which is the maximum and the point is thus non-singular. If we are interested in the position and orientation of a six joint manipulator then the jacobian is a square matrix. In such cases the condition for a point ϕ to be singular reduces to $\det(\mathbf{J}(\phi)) = 0$; that is, the matrix is singular.

6.3 Numerical Methods

The jacobian of a manipulator also finds applications in various numerical methods, for example, to solve the inverse kinematics. As an example, we will look at a method which is the many-variable extension of the Newton-Raphson method.

For a single variable the Newton-Raphson method is as follows. We wish to solve an equation:-

$$f(x) = 0$$

for some function f. We begin with an initial guess $x^{(0)}$, and then refine this guess using the iteration formula:-

$$x^{(i+1)} = x^{(i)} - \frac{f(x^{(i)})}{\frac{\mathrm{d}}{\mathrm{d}x} f(x^{(i)})}$$

Here the notation, superscript (i), denotes the i^{th} iterate.

This generalizes to many variables quite easily. Suppose we have six equations to solve in six variables:-

$$\begin{aligned}
f_1(\theta_1, \ldots, \theta_6) &= 0 \\
f_2(\theta_1, \ldots, \theta_6) &= 0 \\
&\vdots \\
f_6(\theta_1, \ldots, \theta_6) &= 0
\end{aligned}$$

We may summarize this with the vector notation as $\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$. Taylor's theorem tells us that:-

$$\mathbf{f}(\boldsymbol{\theta}+\mathbf{h}) \approx \mathbf{f}(\boldsymbol{\theta}) + \mathbf{J}(\boldsymbol{\theta})\mathbf{h}$$

Now if we assume that $\boldsymbol{\theta}$ is the root we are looking for, then since $\mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$, we can approximate the error $\mathbf{h}$ as:-

$$\mathbf{h} \approx \mathbf{J}^{-1}(\boldsymbol{\theta})\mathbf{f}(\boldsymbol{\theta}+\mathbf{h})$$

Since at this stage, we do not know $\boldsymbol{\theta}$ we cannot calculate $\mathbf{J}^{-1}(\boldsymbol{\theta})$, so we approximate it by $\mathbf{J}^{-1}(\boldsymbol{\theta}+\mathbf{h})$ which is our guess. By setting $\mathbf{h}^{(i)} = \boldsymbol{\theta}^{(i)} - \boldsymbol{\theta}^{(i+1)}$ we can set up the following iterative scheme:-

$$\boldsymbol{\theta}^{(i+1)} = \boldsymbol{\theta}^{(i)} - \mathbf{J}^{-1}(\boldsymbol{\theta}^{(i)})\mathbf{f}(\boldsymbol{\theta}^{(i)})$$

This is the Newton-Raphson formula for many variables. In practice, however, inverting matrices is very slow. A quicker method is to solve the linear equations $\mathbf{J}(\boldsymbol{\theta}^{(i)})\mathbf{h}^{(i)} = \mathbf{f}(\boldsymbol{\theta}^{(i)})$, using Gauss elimination, for example.

To see how this could be used, we look at a simple example. Consider the planar manipulator once more. This time we want to take account of the output angle $\Phi = \theta_1 + \theta_2 + \theta_3$ as well as the position of the end point. We will assume that $l_1 = 2, l_2 = 1$ and $l_3 = 1$ in some units. Suppose the arm is in the position illustrated in fig. 6.1, where $\theta_1 = \pi/3, \theta_2 = \pi/6$ and $\theta_3 = \pi/6$.

The forward kinematics gives the starting position as:-

$$\begin{aligned} x &= 2\cos(\frac{\pi}{3}) + \cos(\frac{\pi}{3}+\frac{\pi}{6}) + \cos(\frac{\pi}{3}+\frac{\pi}{6}+\frac{\pi}{6}) &&= 0.5000 \\ y &= 2\sin(\frac{\pi}{3}) + \sin(\frac{\pi}{3}+\frac{\pi}{6}) + \sin(\frac{\pi}{3}+\frac{\pi}{6}+\frac{\pi}{6}) &&= 3.5981 \\ \Phi &= \frac{\pi}{3}+\frac{\pi}{6}+\frac{\pi}{6} &&= 2\frac{\pi}{3} \end{aligned}$$

Now suppose we want to move the end-effector to the position where:-

$$x = 0.5, \qquad y = 3.0, \qquad \Phi = \frac{2\pi}{3}$$

We set up the three functions:-

$$\begin{aligned} f_1 &= 2\cos(\theta_1) + \cos(\theta_1+\theta_2) + \cos(\theta_1+\theta_2+\theta_3) - 0.5 \\ f_2 &= 2\sin(\theta_1) + \sin(\theta_1+\theta_2) + \sin(\theta_1+\theta_2+\theta_3) - 3.0 \\ f_3 &= \theta_1+\theta_2+\theta_3 - \frac{2\pi}{3} \end{aligned}$$

In the desired position all three of these functions will vanish. So we may use the Newton-Raphson method to find the roots, that is the values of the joint angles. The jacobian has columns:-

$$\mathbf{J}(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} -2\sin\theta_1 - \sin(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2+\theta_3) \\ -\sin(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & \\ 2\cos\theta_1 + \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2+\theta_3) \\ +\cos(\theta_1+\theta_2+\theta_3) & +\cos(\theta_1+\theta_2+\theta_3) & \\ 1 & 1 & 1 \end{pmatrix}$$

As our initial guess we may as well use the starting position, so that $\boldsymbol{\theta}^{(0)} = (\pi/3, \pi/6, \pi/6)^T$. So now:-

$$\mathbf{J}(\boldsymbol{\theta}^{(0)}) = \begin{pmatrix} -3.5981 & -1.8660 & -0.8660 \\ 0.5000 & -0.5000 & -0.5000 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\boldsymbol{\theta}^{(0)}) = \begin{pmatrix} 0.0000 \\ 0.5981 \\ 0.0000 \end{pmatrix}$$

We find the first approximation to the error by solving:-

$$\begin{pmatrix} -3.5981 & -1.8660 & -0.8660 \\ 0.5000 & -0.5000 & -0.5000 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \\ h_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0.0000 \\ 0.5981 \\ 0.0000 \end{pmatrix}$$

To four decimal places the solution is:-

$$\begin{pmatrix} h_1^{(0)} \\ h_2^{(0)} \\ h_3^{(0)} \end{pmatrix} = \begin{pmatrix} 0.5981 \\ -1.6340 \\ 1.0359 \end{pmatrix} \quad \text{and thus} \quad \boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(0)} - \mathbf{h}^{(0)} = \begin{pmatrix} 0.4491 \\ 2.1576 \\ -0.5123 \end{pmatrix}$$

For the next iteration the values of the jacobian and the functions are:-

$$\mathbf{J}(\boldsymbol{\theta}^{(1)}) = \begin{pmatrix} 2.2441 & -1.3758 & -0.8660 \\ 0.4413 & -1.4603 & -0.5000 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{f}(\boldsymbol{\theta}^{(1)}) = \begin{pmatrix} -0.059 \\ -0.756 \\ 0.0000 \end{pmatrix}$$

This then gives:-

$$\mathbf{h}^{(1)} = \begin{pmatrix} -0.1606 \\ 0.5493 \\ -0.3887 \end{pmatrix} \quad \text{and therefore} \quad \boldsymbol{\theta}^{(2)} = \begin{pmatrix} 0.6097 \\ 1.6083 \\ -0.1236 \end{pmatrix}$$

The values of the functions here are now $\mathbf{f}(\boldsymbol{\theta}^{(2)}) = (0.0367, -0.1910, 0.0000)^T$, which is getting closer to zero. The next two iterations give:-

$$\boldsymbol{\theta}^{(3)} = \begin{pmatrix} 0.6789 \\ 1.4106 \\ 0.0049 \end{pmatrix}, \qquad \mathbf{f}(\boldsymbol{\theta}^{(3)}) = \begin{pmatrix} 0.0608 \\ -0.0096 \\ 0.0000 \end{pmatrix}$$

and

$$\boldsymbol{\theta}^{(4)} = \begin{pmatrix} 0.6984 \\ 1.4329 \\ -0.0369 \end{pmatrix}, \qquad \mathbf{f}(\boldsymbol{\theta}^{(4)}) = \begin{pmatrix} 0.0001 \\ -0.0010 \\ 0.0000 \end{pmatrix}$$

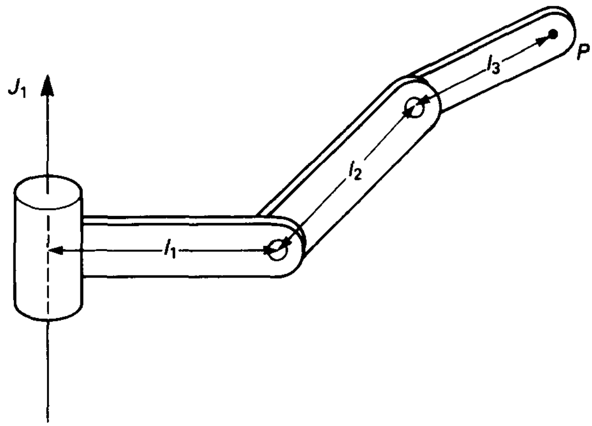


Figure 6.2 A Three Joint Manipulator

If we can tolerate an accuracy of only two decimal places we can stop here. Otherwise we could continue to any desired accuracy. In this particular case we have an exact solution of the inverse kinematics: compare the results here with those of exercise 5.3.

Exercises

6.1 A manipulator has the kinematic structure illustrated in fig. 6.2.

(i) By setting up a suitable co-ordinate system and home position, find the kinematic equations for the co-ordinates of the point P.

(ii) Calculate the jacobian of this manipulator.

(iii) Show that the limiting positions, where the determinant of the jacobian vanishes, lie on the surface of a hollow torus. Assume that $l_1 > l_2 + l_3$.

(iv) How many postures are there in general?

(v) If $l_1 < (l_2 + l_3)$, show that there exist points with four postures, and that points on the J_1 axis have a continuous set of postures.

6.2 For a parallel manipulator it is the inverse kinematics that gives a mapping, this time from the space of rigid body motions to joint space. Find the jacobian matrix for the parallel planar manipulator whose inverse kinematics were found in exercise 5.7.

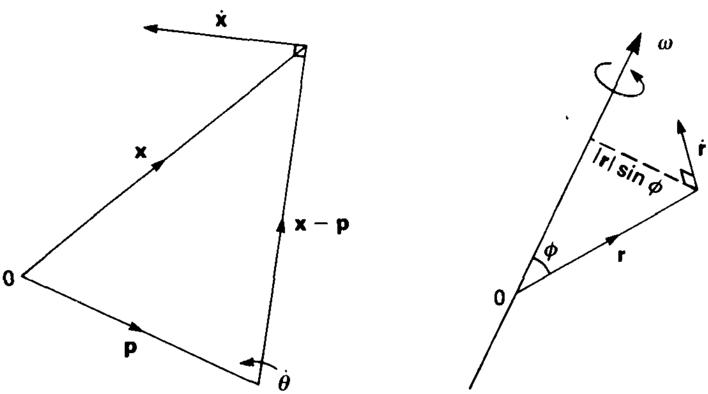


Figure 6.3 (a) Linear Velocity Given by an Instantaneous Centre of Rotation (b) Relation Between Angular and Linear Velocities

6.4 Linear Velocities

Perhaps the most important use for jacobians is for relating the joint velocities to the link velocities. In section 6.1 we saw that $\Delta x \approx \mathbf{J}\,\Delta\theta$. Dividing by Δt and proceeding to the limit we obtain the exact relation:-

$$\dot{\mathbf{x}} = \mathbf{J}\,\dot{\theta}$$

The dots, as usual, denote differentiation with respect to time. This is quite general, but usually we are interested in the linear velocity of some point on a link, or the angular velocity of a link. The movements that can be performed by robots are very general; however, for any rigid body motion in the plane there is always a centre of rotation. Similarly for motion on the surface of a sphere, as one gets from a spherical wrist, there is always an axis of rotation. For rigid movements in three dimensions there is always a fixed line; the screw axis. If a rigid body undergoes some complicated motion in the plane, for example, then at any time in the body's motion there will be an instantaneous centre of rotation. Similarly we get instantaneous rotation axes and instantaneous screw axes. As we shall see below, these concepts are closely related to the velocities that we are interested in.

In two dimensions we have a simple relation between the velocity of a point and the instantaneous centre of rotation, see fig. 6.3(a). This can also be shown using the 3×3 matrices which represent rigid movements. Let p be the centre of rotation and suppose that we wish to know the velocity of the point x. Now the position of x is given by:-

$$\begin{pmatrix} \mathbf{x}(t) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}(\theta) & (\mathbf{I} - \mathbf{R}(\theta))\mathbf{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix}$$

Note that from now on we will not write the partition lines in partitional matrices. Now assume that $\theta = 0$ when $t = 0$. We can always arrange for this to be true by beginning the measurements from the point we are interested in. Now at $\theta = 0$ the time derivative of the above is given by:-

$$\begin{pmatrix} \dot{\mathbf{x}}(0) \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{R}}(0)\dot{\theta} & -\dot{\mathbf{R}}(0)\mathbf{p}\dot{\theta} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix}$$

This gives the equation:-

$$\dot{\mathbf{x}}(0) = \dot{\mathbf{R}}(0)(\mathbf{x}(0) - \mathbf{p})\dot{\theta}$$

But we know that $\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so taking the differential and setting $\theta = 0$ gives:-

$$\dot{\mathbf{R}}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

As mentioned above, we may begin measuring time anywhere, so these results apply for any time, not just $t = 0$. We can drop the time dependence and write:-

$$\dot{x} = (p_y - y)\dot{\theta}$$
$$\dot{y} = (x - p_x)\dot{\theta}$$

Notice that the vector $\dot{\mathbf{x}}$ is always normal to $(\mathbf{x} - \mathbf{p})$. These results can be used to compute the jacobian of planar manipulators. For a three joint planar manipulator we have:-

$$\begin{pmatrix} \mathbf{x}(t) \\ 1 \end{pmatrix} = \mathbf{A}_1(\theta_1)\mathbf{A}_2(\theta_2)\mathbf{A}_3(\theta_3) \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix}$$

Again we can arrange things so that at the point of interest $\theta_1 = \theta_2 = \theta_3 = 0$. Then since $\mathbf{A}_i(0)$ is the identity matrix, when we differentiate and set the joint angles to zero, we get:-

$$\begin{pmatrix} \dot{\mathbf{x}}(0) \\ 0 \end{pmatrix} = \dot{\mathbf{A}}_1(0) \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix} \dot{\theta}_1 + \dot{\mathbf{A}}_2(0) \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix} \dot{\theta}_2 + \dot{\mathbf{A}}_3(0) \begin{pmatrix} \mathbf{x}(0) \\ 1 \end{pmatrix} \dot{\theta}_3$$

Once again there is nothing special about the point $\theta_1 = \theta_2 = \theta_3 = 0$, and our result applies quite generally. For each $\mathbf{A}$-matrix the centre of rotation is simply the current position of the joint. So if we denote the current position of joint i by $\mathbf{j}_i$ we have:-

$$\dot{x} = (j_{1y} - y)\dot{\theta}_1 + (j_{2y} - y)\dot{\theta}_2 + (j_{3y} - y)\dot{\theta}_3$$
$$\dot{y} = (x - j_{1x})\dot{\theta}_1 + (x - j_{2x})\dot{\theta}_2 + (x - j_{3x})\dot{\theta}_3$$

This can be neatly summarized as:-

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} (j_{1y} - y) & (j_{2y} - y) & (j_{3y} - y) \\ (x - j_{1x}) & (x - j_{2x}) & (x - j_{3x}) \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}$$

And this shows us that the jacobian is given by:-

$$\mathbf{J}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} (j_{1y} - y) & (j_{2y} - y) & (j_{3y} - y) \\ (x - j_{1x}) & (x - j_{2x}) & (x - j_{3x}) \end{pmatrix}$$

The suitably altered co-ordinates of the joints are the columns of the jacobian.

The planar manipulator in section 6.1 had its joints at:-

$$\begin{pmatrix} j_{1x} \\ j_{1y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} j_{2x} \\ j_{2y} \end{pmatrix} = \begin{pmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \end{pmatrix},$$

$$\begin{pmatrix} j_{3x} \\ j_{3y} \end{pmatrix} = \begin{pmatrix} l_1 \cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2\sin(\theta_1+\theta_2) \end{pmatrix}$$

and the end point has co-ordinates:-

$$\begin{aligned} x &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ y &= l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \end{aligned}$$

So the jacobian is exactly as calculated in section 6.1. Its columns are:-

$$\begin{aligned} \begin{pmatrix} \frac{\partial x}{\partial\theta_1} \\ \frac{\partial y}{\partial\theta_1} \end{pmatrix} &= \begin{pmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \end{pmatrix} \\ \begin{pmatrix} \frac{\partial x}{\partial\theta_2} \\ \frac{\partial y}{\partial\theta_2} \end{pmatrix} &= \begin{pmatrix} l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \end{pmatrix} \\ \begin{pmatrix} \frac{\partial x}{\partial\theta_3} \\ \frac{\partial y}{\partial\theta_3} \end{pmatrix} &= \begin{pmatrix} l_3\sin(\theta_1+\theta_2+\theta_3) \\ l_3\cos(\theta_1+\theta_2+\theta_3) \end{pmatrix} \end{aligned}$$

but here we have not had to find any derivatives.

6.5 Angular Velocities

The angular velocity of a rigid body is a vector. It is aligned along the instantaneous rotation axis of the body and its magnitude is the angular speed about the axis. Consider a point $\mathbf{r}$ attached to a body rotating with angular velocity $\boldsymbol{\omega}$; see fig. 6.3(b). The linear velocity of the point is given by $\dot{\mathbf{r}} = \boldsymbol{\omega} \wedge \mathbf{r}$. If we represent the rotations by 3×3 matrices we have that:-

$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}(0)$$

Differentiating and setting $t = 0$ gives:-

$$\dot{\mathbf{r}}(0) = \dot{\mathbf{R}}(0)\mathbf{r}(0)$$

Comparing this with our first result shows that $\dot{\mathbf{R}}(0)$ must have the same effect on vectors as '$\boldsymbol{\omega}\wedge$'; in other words for any vector $\mathbf{a}$ we must have:-

$$\dot{\mathbf{R}}(0)\mathbf{a} = \boldsymbol{\omega} \wedge \mathbf{a}$$

This is not hard to solve, see exercise 2.7. It gives us that:-

$$\dot{\mathbf{R}}(0) = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$$

This can be used to find the velocity of the last link of a spherical wrist. For a three joint wrist the overall transformation $\mathbf{K}$ is the product of three rotations, $\mathbf{K} = \mathbf{R}_1(\theta_1)\mathbf{R}_2(\theta_2)\mathbf{R}_3(\theta_3)$. The derivative when all the joint angles are zero is:-

$$\dot{\mathbf{K}} = \dot{\mathbf{R}}_1 + \dot{\mathbf{R}}_2 + \dot{\mathbf{R}}_3$$

and hence the angular velocity of the final link is just the sum of the angular velocities of the joints. Now, because each joint just rotates about its axis, we can write the angular velocities of the joints as $\omega_i = \hat{\mathbf{v}}_i\dot{\theta}_i$; where $\hat{\mathbf{v}}_i$ is the unit vector along the i^{th} joint. The angular velocity of the last link can be expressed by the following matrix equation:-

$$\boldsymbol{\omega} = \begin{pmatrix} \hat{v}_{1x} & \hat{v}_{2x} & \hat{v}_{3x} \\ \hat{v}_{1y} & \hat{v}_{2y} & \hat{v}_{3y} \\ \hat{v}_{1z} & \hat{v}_{2z} & \hat{v}_{3z} \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{pmatrix}$$

Notice that the columns of the jacobian here are just the vectors along the joint axes. So it is easy now to calculate the jacobian of the 3-R wrist, introduced in section 4.2, for example.

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\mathbf{v}}_2 = \mathbf{R}(\theta_1, \mathbf{k}) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ 0 \end{pmatrix},$$

$$\hat{\mathbf{v}}_3 = \mathbf{R}(\theta_1, \mathbf{k})\mathbf{R}(\theta_2, \mathbf{j}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}$$

Hence the jacobian for this manipulator is:-

$$\mathbf{J} = \begin{pmatrix} 0 & -\sin\theta_1 & \cos\theta_1 \sin\theta_2 \\ 0 & \cos\theta_1 & \sin\theta_1 \sin\theta_2 \\ 1 & 0 & \cos\theta_2 \end{pmatrix}$$

6.6 Combining Linear and Angular Velocities

For a rigid body moving in three dimensions we want to know both its angular velocity and the linear velocity of its points. We can find these by considering a general screw motion:-

$$\begin{pmatrix} \mathbf{R} & \mathbf{t} \\ 0 & 1 \end{pmatrix}$$

In section 2.6 we saw that the translation vector is given by $\mathbf{t} = \frac{\theta p}{2\pi}\hat{\mathbf{v}} + (\mathbf{I} - \mathbf{R})\mathbf{u}$, where p is the pitch of the screw, $\hat{\mathbf{v}}$ a unit vector along its axis, $\mathbf{u}$ a point on the axis and we have used θ, rather than ϕ, for the joint variable which depends on time. The velocity of a point $\mathbf{x}$ is then given by:-

$$\begin{pmatrix} \dot{\mathbf{x}} \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\mathbf{R}} & \dot{\mathbf{t}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}$$

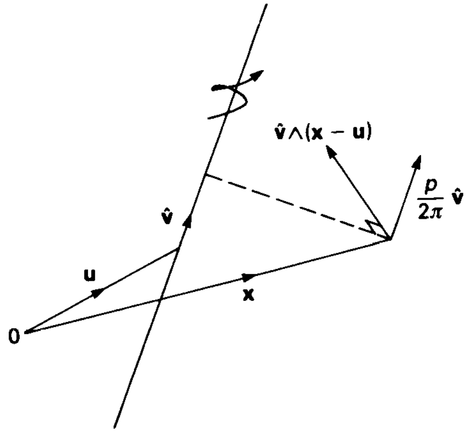


Figure 6.4 A Screw Motion

Using what we already know about the derivatives of rotation matrices, this can be written as:-

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \wedge \mathbf{x} + \mathbf{s}$$

The linear velocity $\mathbf{s} = \dot{\mathbf{t}}$ is a characteristic velocity of the motion. Physically it is the linear velocity of points on a line through the origin parallel to the axis of rotation. We can find $\mathbf{s}$ by differentiating $\mathbf{t}$:-

$$\mathbf{s} = \dot{\mathbf{t}} = \dot{\theta}\frac{p}{2\pi}\hat{\mathbf{v}} - \dot{\mathbf{R}}\,\mathbf{u} = \dot{\theta}\frac{p}{2\pi}\hat{\mathbf{v}} - \boldsymbol{\omega} \wedge \mathbf{u}$$

Notice that the term $\dot{\theta}\frac{p}{2\pi}\hat{\mathbf{v}}$ is the velocity of a point lying on the screw axis. So, as we would expect from contemplating fig. 6.4, the velocity of a point $\mathbf{x}$ can be written more fully as:-

$$\dot{\mathbf{x}} = \boldsymbol{\omega} \wedge (\mathbf{x} - \mathbf{u}) + \dot{\theta}\frac{p}{2\pi}\hat{\mathbf{v}}$$

We can combine the angular and linear velocities into six component vectors $\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{s} \end{pmatrix}$. These six component vectors are called **instantaneous screws**. As we shall see, they are for rigid bodies the analogue of the angular velocity of particles.

Now, if the screw motion is about a joint, then $\hat{\mathbf{v}}$ is the unit vector in the direction of the joint, $\mathbf{u}$ is the position vector of a point on the joint axis and the angular velocity will be $\boldsymbol{\omega} = \hat{\mathbf{v}}\dot{\theta}$. Finally p is the pitch of the joint. The velocity of a point attached to the joint will be given by:-

$$\dot{\mathbf{x}} = (\hat{\mathbf{v}} \wedge (\mathbf{x} - \mathbf{u}) + \frac{p}{2\pi}\hat{\mathbf{v}})\dot{\theta}$$

Connecting six joints together, as in a serial robot, both the linear and angular velocities add vectorially to give the angular velocity $\boldsymbol{\omega}$, and the linear velocity $\mathbf{s}$, of the last link. So we may write this in terms of instantaneous screws:-

$$\begin{pmatrix} \omega \\ \mathbf{s} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{v}}_1 \\ \mathbf{u}_1 \wedge \hat{\mathbf{v}}_1 + \frac{p_1}{2\pi}\hat{\mathbf{v}}_1 \end{pmatrix}\dot{\theta}_1 + \begin{pmatrix} \hat{\mathbf{v}}_2 \\ \mathbf{u}_2 \wedge \hat{\mathbf{v}}_2 + \frac{p_2}{2\pi}\hat{\mathbf{v}}_2 \end{pmatrix}\dot{\theta}_2 + \cdots + \begin{pmatrix} \hat{\mathbf{v}}_6 \\ \mathbf{u}_6 \wedge \hat{\mathbf{v}}_6 + \frac{p_1}{2\pi}\hat{\mathbf{v}}_6 \end{pmatrix}\dot{\theta}_6$$

This can be condensed into the matrix equation:-

$$\begin{pmatrix} \omega \\ \mathbf{s} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{pmatrix}$$

The columns of the jacobian matrix $\mathbf{J}$ are given by $\begin{pmatrix} \hat{\mathbf{v}}_i \\ \mathbf{u}_i \wedge \hat{\mathbf{v}}_i + \frac{p_i}{2\pi}\hat{\mathbf{v}}_i \end{pmatrix}$, and are determined by the i^{th} joint. In other words, to each joint there is an associated instantaneous screw which depends only on the position, orientation and pitch of the joint. These 'joint screws' are reasonably easy to calculate and once again we have been able to find the jacobian matrix of the manipulator without computing any partial derivatives.

The jacobian is of fundamental importance in robotics. This is because it is the linearization of the forward kinematics. Hence, it tells us about errors, velocities and other first order properties of the robot. For robot manipulators with an open loop structure the jacobian is very simple to calculate since its columns are given by the joints of the robot.

Exercises

6.3 In two dimensions a sliding joint is represented by a matrix $\begin{pmatrix} \mathbf{I} & \mathbf{x} \\ 0 & 1 \end{pmatrix}$, where $\mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} d$, with $d = d(t)$ is a function of time.

(i) Find the velocity of a point undergoing such a translation. If such a joint were used in a serial manipulator, what would be the corresponding column in the jacobian?

(ii) A planar manipulator consists of a revolute joint and a sliding joint. In the home configuration the revolute joint is at the origin, the sliding joint is aligned along the x-axis and a point Q attached to the last link has co-ordinates $(1, 0)$. Find the velocity of Q as a function of the joint angle θ of the revolute joint and the extension d of the sliding joint.

6.4 Find the jacobian of the Cincinnati T^3 wrist illustrated in fig. 4.4.

6.5 Let $\mathbf{R}(t)$ be a one parameter family of rotation matrices, such that $\mathbf{R}(0) = \mathbf{I}$. Using the fact that rotation matrices satisfy $\mathbf{R}\mathbf{R}^T = \mathbf{I}$, show that $\dot{\mathbf{R}}(0)$ is an antisymmetric matrix.

6.6 The instantaneous screw of a revolute joint is given by $\begin{pmatrix} \hat{\mathbf{v}} \\ \mathbf{u} \wedge \hat{\mathbf{v}} \end{pmatrix}$. If a rotation $\mathbf{R}$ followed by a translation $\mathbf{t}$ is performed on the joint, show that the new joint will have the instantaneous screw given by:-

$$\begin{pmatrix} \mathbf{R} & 0 \\ \mathbf{TR} & \mathbf{R} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{v}} \\ \mathbf{u} \wedge \hat{\mathbf{v}} \end{pmatrix}$$

where $\mathbf{T} = \begin{pmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{pmatrix}$.