# 5 Inverse Kinematics

In the last chapter we saw how to derive the kinematics of a serial robot. The position and orientation of any point rigidly attached to the gripper can be found if the joint angles are known. In this section we want to do the reverse. Given the position and orientation of the gripper required, to what angles must the joints be set? This is one of the central problems in robotics, since whenever we specify the motion of the robot's gripper we need to know the corresponding joint motions. Essentially we must solve the following matrix equation:-

$$\mathbf{A}_1(\theta_1)\mathbf{A}_2(\theta_2)\mathbf{A}_3(\theta_3)\mathbf{A}_4(\theta_4)\mathbf{A}_5(\theta_5)\mathbf{A}_6(\theta_6) = \mathbf{K}$$

where $\mathbf{K}$ is the constant matrix which specifies the position and orientation of the gripper. This constitutes a set of highly non-linear equations for the joint angles $\theta_1, \theta_2, \dots, \theta_6$.

In general, very little is known about solving such equations; even the number of solutions is problematic. For $n$ non-linear equations in $n$ unknowns there may be no solutions at all, one or more discrete solutions or even continuous families of solutions. This contrasts sharply with the case of linear equations, where only a single solution or a linear space of solutions is possible. In the linear case we can look to the determinant of the system to distinguish these cases; for non-linear equations no such test exists.

Things are not quite so bad if there are no helical joints, since the joint angles only appear in the equations as $\cos\theta_i$ or $\sin\theta_i$. Now if we use these as our variables the equations are **algebraic**. That is, they are only polynomials in the variables $\cos\theta_i$ and $\sin\theta_i$. So if we solve for these variables it is a simple matter to find the joint angles; $\theta_i$. However, we have actually doubled the number of variables in the equations but we must also consider the relations between the new variables. This means we must include the equations:-

$$\cos^2\theta_i + \sin^2\theta_i = 1$$

in our non-linear system.

There is another technique used to make the equations algebraic: to write the equations in terms of 'tan half angles'; that is, to make the substitutions:-

$$\cos\theta_i = \frac{1 - t_i^2}{1 + t_i^2} \qquad \sin\theta_i = \frac{2t_i}{1 + t_i^2}$$

where $t_i = \tan(\theta_i/2)$. The only disadvantage of this approach is that it fails when $\theta_i = \pi$.

Algebraic equations have nice properties. For example a polynomial equation in one variable has as many solutions (roots) as the degree of the polynomial. This is familiar from elementary algebra, and also we recall that the roots must be counted properly; repeated roots and complex roots must be accounted for. There is a generalization of this to systems of polynomial equations in several variables. If we have $n$ equations of degree $d_1, d_2, \ldots, d_n$ in $n$ unknowns then, in general, we get $d_1 \times d_2 \times \ldots \times d_n$ solutions. However, there are exceptional circumstances when there is an infinite family of solutions.

So, for example, consider two quadratics in two variables. Quadratics in two variables are just conic curves; ellipses, parabolas and hyperbolas. Their degree is two, so two of them should intersect in $2 \times 2 = 4$ points. Some of these intersections may be complex; they will occur in complex conjugate pairs if the coefficients of the equations are real. Hence there may be no real intersections at all. Singular solutions are also possible. They correspond to repeated roots in the one variable case, and occur when the curves intersect and have the same tangent at the intersection. See fig. 5.1.

## 5.1 The Planar Manipulator

To get back to the problem of inverse kinematics let us look at a simple example. The planar manipulator exhibits all the possibilities that can arise. Consider the position after just two links, see fig. 5.2. The kinematic equations for the end point are:-

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{aligned}$$

Given $x$ and $y$ we must find $\cos\theta_1, \sin\theta_1, \cos\theta_2$ and $\sin\theta_2$. The above equations are in fact quadratic, since we can use the trigonometric formulas to write:-

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \qquad \sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

Then together with the identities satisfied by the sine and cosine functions, these give us four quadratic equations in four unknowns:-

$$\begin{aligned} x &= l_1\cos\theta_1 + l_2\cos\theta_1\cos\theta_2 - l_2\sin\theta_1\sin\theta_2 && (A) \\ y &= l_1\sin\theta_1 + l_2\sin\theta_1\cos\theta_2 + l_2\cos\theta_1\sin\theta_2 && (B) \\ 1 &= \cos^2\theta_1 + \sin^2\theta_1 && (C) \\ 1 &= \cos^2\theta_2 + \sin^2\theta_2 && (D) \end{aligned}$$

So we might expect $2 \times 2 \times 2 \times 2 = 16$ solutions: in fact only 2 arise. The discrepancy is accounted for by four singular complex solutions at 'infinity'.

To solve this system we square equation $(A)$ and add it to the square of $(B)$:-

$$\begin{aligned} (x^2 + y^2) &= l_1^2(\cos^2\theta_1 + \sin^2\theta_1) + l_2^2(\cos^2\theta_1 + \sin^2\theta_1)\cos^2\theta_2 \\ &\quad + l_2^2(\cos^2\theta_1 + \sin^2\theta_1)\sin^2\theta_2 + 2l_1l_2(\cos^2\theta_1 + \sin^2\theta_1)\cos\theta_2 \end{aligned}$$

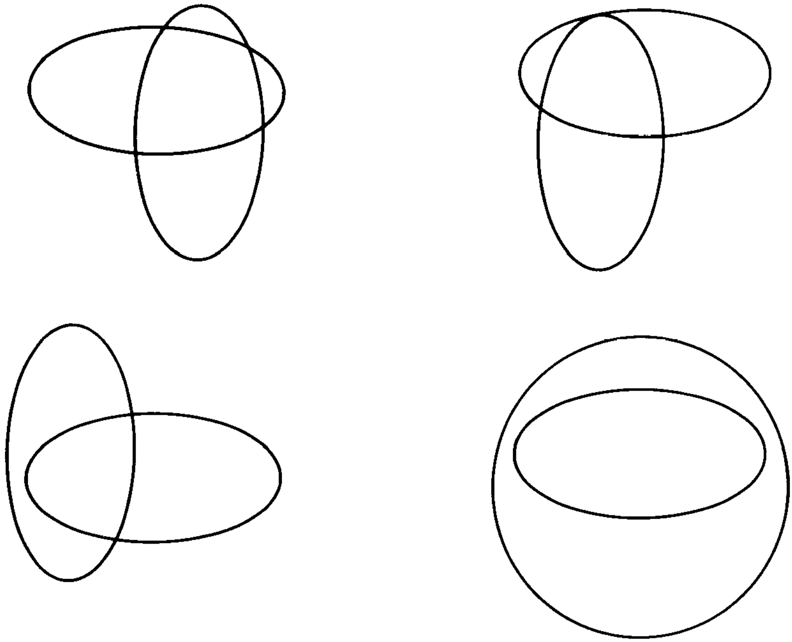

**Figure 5.1** Some Possible Intersections of Two Conics

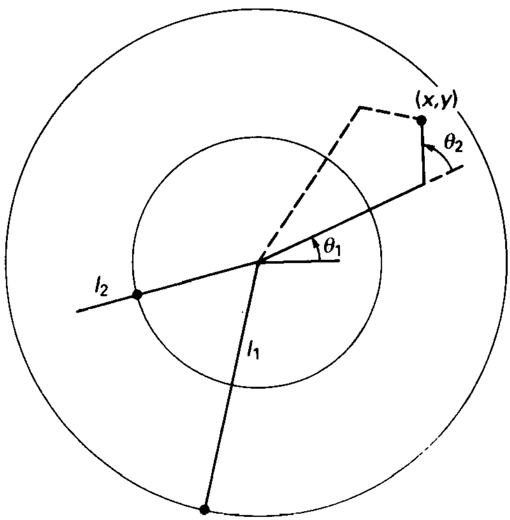


**Figure 5.2** The Planar Manipulator; Postures and Work Space

Then using $(C)$ and $(D)$ to simplify we obtain:-

$$(x^2 + y^2) = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2$$

In fact this is just the cosine rule from trigonometry. So the solution for $\cos\theta_2$ is just:-

$$\cos\theta_2 = \frac{1}{2l_1 l_2}\{(x^2 + y^2) - (l_1^2 + l_2^2)\} = \lambda$$

We abbreviate this to $\lambda$ since it will occur frequently. Hence, by $(D)$ $\sin\theta_2$ is:-

$$\sin\theta_2 = \pm(1 - \lambda^2)^{\frac{1}{2}}$$

that is, there are two possible solutions. We will look at this in more detail in a moment. First let us find $\cos\theta_1$ and $\sin\theta_1$. The simplest way to do this is to form the two equations:-

$$\begin{aligned}(A)\cos\theta_1 + (B)\sin\theta_1 &\equiv x\cos\theta_1 + y\sin\theta_1 = l_1 + l_2\cos\theta_2\\ &= l_1 + l_2\lambda\end{aligned}$$

$$\begin{aligned}-(A)\sin\theta_1 + (B)\cos\theta_1 &\equiv -x\sin\theta_1 + y\cos\theta_1 = l_2\sin\theta_2\\ &= \pm l_2(1 - \lambda^2)^{\frac{1}{2}}\end{aligned}$$

Again, the relation $(C)$ has been used to simplify the above. Now we have two simultaneous linear equations which are easily solved. So we have found explicit equations for the sines and cosines of the angles in terms of the design parameters and the position of the manipulator's end-effector. In fact there are two solutions, corresponding to the upper and lower sign choices. These equations are the inverse kinematic relations for the manipulator:-

$$\begin{aligned}\cos\theta_1 &= \frac{1}{(x^2 + y^2)}\{x(l_1 + l_2\lambda) \pm y l_2(1 - \lambda^2)^{\frac{1}{2}}\}\\ \sin\theta_1 &= \frac{1}{(x^2 + y^2)}\{\mp x l_2(1 - \lambda^2)^{\frac{1}{2}} + y(l_1 + l_2\lambda)\}\\ \cos\theta_2 &= \frac{1}{2l_1 l_2}\{(x^2 + y^2) - (l_1^2 + l_2^2)\} = \lambda\\ \sin\theta_2 &= \pm(1 - \lambda^2)^{\frac{1}{2}}\end{aligned}$$

## 5.2 Postures

For the planar manipulator of the previous section there are generally two solutions for the inverse kinematics. They arise from the sign of the term $\sin\theta_2$: physically this corresponds to the fact that there are two ways of reaching any point in the plane, see fig. 5.2. These two configurations of the manipulator are called **postures**; one is referred to as 'elbow up', the other as 'elbow down'. However, not every point $(x, y)$ has two postures. There is only one solution for $\sin\theta_2$ if $\sin\theta_2 = 0$; that is when $\lambda = \pm 1$, which corresponds to $\theta_2 = 0$ or

$\pi$. The points in the plane determined by these values are given by:-

$$\begin{array}{lll} \cos\theta_2 = 1; & \text{gives} & (x^2+y^2) = (l_1+l_2)^2 \\ \cos\theta_2 = -1; & \text{gives} & (x^2+y^2) = (l_1-l_2)^2 \end{array}$$

These are the equations of two concentric circles, on one the arm is at full stretch, while to reach the other the arm must double back on itself, see fig. 5.2.

Beyond the outer circle and inside the smaller one, the solutions for $\sin\theta_2$ become complex and it is clear that we cannot reach such points with a real arm. The annular region is the projection of the robot's **work space** onto the plane, see section 3.3. It is the space that the robot can reach and work in. The work space of any robot is always bounded by curves or surfaces on which the number of postures is different from the body of the work space. Such points are called **singular points**; however, singular points may also occur in the interior of the work space. A better characterization of singular points is points where the robot loses one or more degrees-of-freedom. In the case of the planar manipulator it is easy to see that on the boundary of the work space the arm has no freedom to move in a radial direction.

So far we have said nothing about the design parameters $l_1$ and $l_2$. In fact the relative sizes of the links do not affect the number of postures, except in the very special case that $l_1 = l_2$. In this case there are still generally two postures for every point in the work space, but the inner boundary has now shrunk to a point; the origin. If we try to place the tip of the manipulator at $x = 0$, $y = 0$, then certainly we must have $\cos\theta_2 = -1$ and $\sin\theta_2 = 0$, but our method for finding $\theta_1$ breaks down. It is quite clear though that there is no restriction on $\theta_1$, so instead of one or two postures, this point in the work space has a whole circle of postures. This kind of singularity, with a continuous family of postures, is particularly difficult to deal with when it comes to controlling the robot. Unfortunately, all six axis robots that have been designed or built have such singularities in their work space. It is not known if this can be avoided.

## 5.3 The 3-R Wrist

As we saw in section 4.2 the kinematic equations of the 3-R wrist can be written in terms of the Euler angles as:-

$$\psi = \theta_1, \qquad \theta = \theta_2, \qquad \phi = \theta_3$$

so there does not seem to be any problem about the inverse kinematics. However, we must remember that the Euler angles have a limited range whilst the joint angles can range over a full circle, 0 to $2\pi$; at least in theory. The kinematic matrix, as we saw in section 4.2, is given by:-

$$\mathbf{K} = \begin{pmatrix} \cos\theta_1\cos\theta_2\cos\theta_3 - \sin\theta_1\sin\theta_3 & -\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\cos\theta_3 & \cos\theta_1\sin\theta_2 \\ \sin\theta_1\cos\theta_2\cos\theta_3 + \cos\theta_1\sin\theta_3 & -\sin\theta_1\cos\theta_2\sin\theta_3 + \cos\theta_1\cos\theta_3 & \sin\theta_1\sin\theta_2 \\ -\sin\theta_2\cos\theta_3 & \sin\theta_2\sin\theta_3 & \cos\theta_2 \end{pmatrix}$$

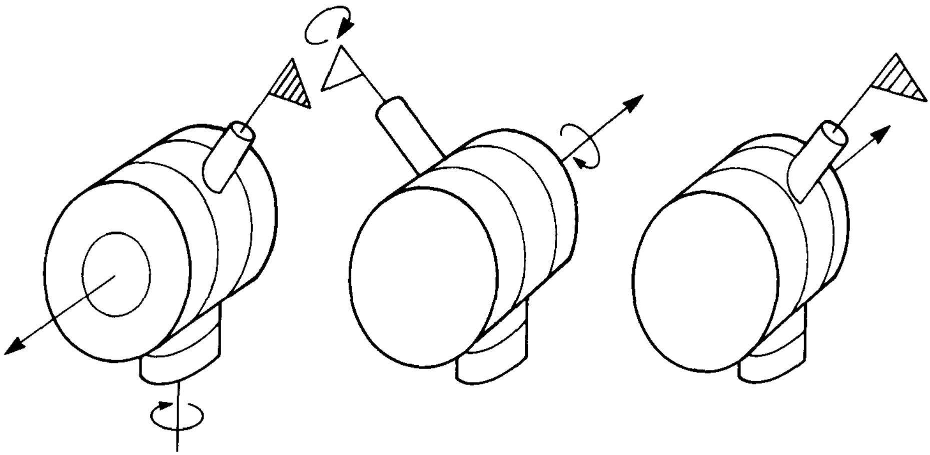

**Figure 5.3** Flip and No Flip Postures of the 3-R Wrist

Notice that we get the same matrix if we make the substitutions:-

$$\theta_1' = \pi + \theta_1, \qquad \theta_2' = 2\pi - \theta_2, \qquad \theta_3' = \pi + \theta_3$$

This is because we can use the usual trigonometric relations to give:-

$$\begin{array}{rclrclrcl} \cos\theta_1' & = & -\cos\theta_1, & \cos\theta_2' & = & \cos\theta_2, & \cos\theta_3' & = & -\cos\theta_3 \\ \sin\theta_1' & = & -\sin\theta_1, & \sin\theta_2' & = & -\sin\theta_2, & \sin\theta_3' & = & -\sin\theta_3 \end{array}$$

So we have two postures, that is two possible solutions for the joint angles given a $3 \times 3$ rotation matrix. In terms of the Euler angles these solutions can be written as:-

$$\begin{array}{rclcl} \theta_1 & = & \psi & \text{or} & \pi + \psi \\ \theta_2 & = & \theta & \text{or} & 2\pi - \theta \\ \theta_3 & = & \phi & \text{or} & \pi - \phi \end{array}$$

These two postures have been given the names 'flip' and 'no flip'. Fig. 5.3 shows how to change from one posture to the other.

The above results have not been derived in a very systematic way: it is difficult to see whether or not there are other solutions. We can repeat the analysis more efficiently by looking at the effect of the kinematic matrix on two points. Suppose $\mathbf{a} = (0, 0, 1)^T$ and $\mathbf{b} = (1, 0, 0)^T$ are the home positions of two points rigidly attached to the gripper. Then rotating about the three wrist joints will take the points to:-

$$\mathbf{a}' = \mathbf{K}(\theta_1, \theta_2, \theta_3)\mathbf{a} \qquad \text{and} \qquad \mathbf{b}' = \mathbf{K}(\theta_1, \theta_2, \theta_3)\mathbf{b}$$

The co-ordinates of these new points are easily calculated:-

$$\mathbf{a}' = \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \sin\theta_2 \\ \sin\theta_1 \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}$$

$$\mathbf{b}' = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \cos\theta_2 \cos\theta_3 - \sin\theta_1 \sin\theta_3 \\ \sin\theta_1 \cos\theta_2 \cos\theta_3 + \cos\theta_1 \sin\theta_3 \\ -\sin\theta_2 \cos\theta_3 \end{pmatrix}$$

The points have been chosen to simplify the calculations as far as possible. For example, the new position vectors of the points are just the first and third rows of the matrix **K**.

From the first point we see that $\cos\theta_2 = z_a$ and hence we can find the sine of the angle $\sin\theta_2 = \pm\sqrt{(1 - z_a^2)}$. The sine and cosine of the first joint angle can now be found from the $x$ and $y$ co-ordinates of this point; $\cos\theta_1 = \pm x_a / \sqrt{(1 - z_a^2)}$ and $\sin\theta_1 = \pm y_a / \sqrt{(1 - z_a^2)}$. To find the third joint angle we must look at the second point, then; $\cos\theta_3 = \mp z_b / \sqrt{(1 - z_a^2)}$. The sine of $\theta_3$ can be found from $x_b$ and $y_b$:-

$$\begin{aligned} \sin\theta_3 &= y_b \cos\theta_1 - x_b \sin\theta_1 \\ &= \pm \left\{ \frac{x_a y_b - x_b y_a}{\sqrt{(1 - z_a^2)}} \right\} \end{aligned}$$

To summarize, the inverse kinematics of the 3-R wrist, in terms of the positions of the two points, is given by:-

$$\begin{aligned} \cos\theta_1 &= \pm\frac{x_a}{\sqrt{(1-z_a^2)}}, & \sin\theta_1 &= \pm\frac{y_a}{\sqrt{(1-z_a^2)}} \\ \cos\theta_2 &= z_a, & \sin\theta_2 &= \pm\sqrt{(1-z_a^2)} \\ \cos\theta_3 &= \mp\frac{z_b}{\sqrt{(1-z_a^2)}}, & \sin\theta_3 &= \pm\left\{\frac{x_a y_b - x_b y_a}{\sqrt{(1-z_a^2)}}\right\} \end{aligned}$$

The two postures are distinguished by the sign of $\sin\theta_2$. Notice that this analysis also tells us where the number of postures is different from two, since, if $\sin\theta_2 = 0$, we cannot divide by this factor and the above analysis fails. In fact these two points with $\theta_2 = 0$ or $\pi$, are singular points each with an infinite number of postures. At these points the first and last axes coincide, thus the final link can be held fixed while the second joint rotates perpendicular to the first and last joints; see also section 2.5.

# Exercises

**5.1** A planar manipulator has link lengths $l_1 = 2$ and $l_2 = 1$ in some units. Use the inverse kinematic equations to find the joint angles which will place the end point at the following positions:-

(i) $x = (\sqrt{3} + \frac{1}{2})$, $y = 1 + \frac{\sqrt{3}}{2}$

(ii) $x = 2,\quad y = 1 + \sqrt{3}$
(iii) $x = \sqrt{2},\quad y = 1 + \sqrt{2}$

**5.2** The two points $(0, 0, 1)$ and $(1, 0, 0)$ are rigidly attached to the gripper of a 3-R wrist. Use the inverse kinematic equations derived in the text to find the joint angles when these points have the co-ordinates:-

(i) $(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ and $(\frac{3}{4}, -\frac{1}{2}, -\frac{\sqrt{3}}{4})$
(ii) $(\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2})$ and $(\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2})$
(iii) $(\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2})$ and $(\frac{1}{8}, -\frac{\sqrt{3}}{8}, -\frac{\sqrt{3}}{4})$

**5.3** Work out the inverse kinematic relations for the three joint planar manipulator studied in section 4.1. If the links' lengths are $l_1 = 2, l_2 = 1$ and $l_3 = 1$ in some system of units, find the possible joint angles which result in the end point having co-ordinates $x = 0.5, y = 3.0$ and output angle $\Phi = 2\pi/3$ radians.

**5.4** Work out the inverse kinematics of the 3-R wrist in terms of the positions of two points rigidly attached to the gripper and where the home co-ordinates are $(0, 0, 1)$ and $(0, 1, 0)$.

## 5.4 The First Three Joints of the Puma

Now we are in a position to calculate the inverse kinematics for the Puma arm. This is possible because the first three joints of the Puma are almost a planar manipulator while the last three are a 3-R wrist. Hence, the problem can be split into two easier pieces. We will express the inverse kinematics in terms of the components of three points rigidly attached to the gripper. In the home position these points will have co-ordinates:-

$$\mathbf{p}_a = \begin{pmatrix} D_3 \\ 0 \\ L_2 + D_4 + 1 \end{pmatrix}, \qquad \mathbf{p}_b = \begin{pmatrix} D_3 + 1 \\ 0 \\ L_2 + D_4 \end{pmatrix}, \qquad \mathbf{p}_c = \begin{pmatrix} D_3 \\ 0 \\ L_2 + D_4 \end{pmatrix}$$

These points have been chosen to make things easy, for example $\mathbf{p}_c$ is the position of the wrist centre. Hence, only movements about the first three joints will affect the position of $\mathbf{p}_c$. Moreover, if we know the position of the wrist centre we can find solutions for the first three joints.

The forward kinematics, or just a consideration of the geometry, gives:-

$$\begin{aligned} x_c &= D_3 \cos\theta_1 + L_2 \sin\theta_1 \sin\theta_2 + D_4 \sin\theta_1 \sin(\theta_2 + \theta_3) \\ y_c &= D_3 \sin\theta_1 - L_2 \cos\theta_1 \sin\theta_2 - D_4 \cos\theta_1 \sin(\theta_2 + \theta_3) \\ z_c &= L_2 \cos\theta_2 + D_4 \cos(\theta_2 + \theta_3) \end{aligned}$$

See fig. 5.4.

Since the second and third joint axes are parallel they behave like a planar manipulator. The first joint simply allows rotation of the plane. So we could write these relations in

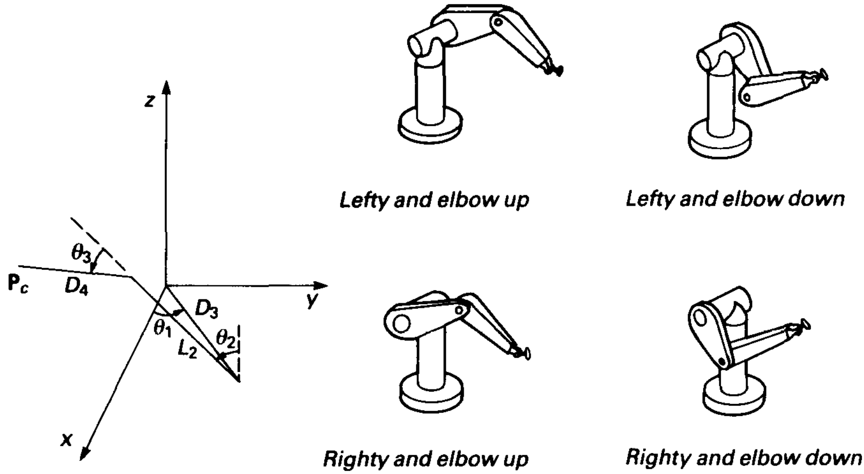


**Figure 5.4** The First Three Joints of the Puma and the Possible Postures

terms of the kinematics of a planar manipulator:-

$$\begin{aligned} x_c &= D_3 \cos\theta_1 - \sin\theta_1 r_y \\ y_c &= D_3 \sin\theta_1 + \cos\theta_1 r_y \\ z_c &= r_z \end{aligned}$$

where $r_y = -L_2 \sin\theta_2 - D_4 \sin(\theta_2 + \theta_3)$ and $r_z = L_2 \cos\theta_2 + D_4 \cos(\theta_2 + \theta_3)$ can be thought of as the forward kinematics of a planar manipulator. The first two equations can be rearranged to give:-

$$\begin{aligned} D_3 &= x_c \cos\theta_1 + y_c \sin\theta_1 \\ r_y &= y_c \cos\theta_1 - x_c \sin\theta_1 \end{aligned}$$

The effect is the same as multiplying by the inverse of $\mathbf{A}_1(\theta_1)$. The first of these new equations is linear in the sine and cosine of $\theta_1$, so we can use it to eliminate $\sin\theta_1$ from the quadratic equation $\cos^2\theta_1 + \sin^2\theta_1 = 1$:-

$$(x_c^2 + y_c^2)\cos^2\theta_1 - 2x_c D_3 \cos\theta_1 + (D_3^2 - y_c^2) = 0$$

Solving this for $\cos\theta_1$ we get:-

$$\cos\theta_1 = \frac{D_3 x_c \pm y_c\sqrt{x_c^2 + y_c^2 - D_3^2}}{(x_c^2 + y_c^2)}$$

using the standard solution for a quadratic. The sine is given by substituting this back in the linear equation:-

$$\sin\theta_1 = \frac{1}{y_c}\left\{ D_3 - x_c \frac{D_3 x_c \pm y_c\sqrt{x_c^2 + y_c^2 - D_3^2}}{(x_c^2 + y_c^2)} \right\}$$

These expressions are going to get very complicated, so to simplify things as much as possible we will write the solutions in terms of the ones we have already found. So it is clear from the above that we could write everything explicitly in terms of the co-ordinates of the points, but we will content ourselves with implicit relations.

Evidently we have two possible solutions for $\theta_1$, depending on the sign of the square root. Each will result in a different posture as we shall see later. Next we use the inverse kinematics of the planar manipulator to solve for $\theta_2$ and $\theta_3$:-

$$\begin{aligned}\cos\theta_3 &= \frac{1}{2L_2D_4}\{(r_y^2 + r_z^2) - (L_2^2 + D_4^2)\} \\ \sin\theta_3 &= \pm\sqrt{1-\cos^2\theta_3}\end{aligned}$$

This introduces a second ambiguity in sign:-

$$\begin{aligned}\cos\theta_2 &= \frac{1}{(r_y^2 + r_z^2)}\{r_z(L_2 + D_4\cos\theta_3) - r_yD_4\sin\theta_3\} \\ \sin\theta_2 &= \frac{1}{(r_y^2 + r_z^2)}\{-r_y(L_2 + D_4\cos\theta_3) + r_zD_4\sin\theta_3\}\end{aligned}$$

These results were simply obtained by substituting $x = r_z, y = -r_y$ in the results of section 5.1.

The two possible sign choices are independent of each other, so there are four possible solutions, and hence four postures. For the Puma these have the cute names 'elbow up' or 'elbow down' depending on the choice of the sign of $\sin\theta_3$, and 'righty' or 'lefty' depending on which sign of $\sqrt{x_c^2 + y_c^2 - D_3^2}$ is chosen. Notice that a little rearrangement gives $\sqrt{x_c^2 + y_c^2 - D_3^2} = (y_c\cos\theta_1 - x_c\sin\theta_1)$. Hence given a set of joint angles, we can tell which posture the robot is in by looking at the sign of these two functions. Including the two possible postures for the wrist, the Puma has eight different postures in all, see fig. 5.4.

## 5.5 The Last Three Joints of the Puma

Most of the hard work here has been done in section 5.3. The only difference is the effect of the first three joints. Remember we are trying to solve the equations:-

$$\mathbf{A}_1(\theta_1)\mathbf{A}_2(\theta_2)\mathbf{A}_3(\theta_3)\mathbf{A}_4(\theta_4)\mathbf{A}_5(\theta_5)\mathbf{A}_6(\theta_6)\mathbf{p} = \mathbf{p}'$$

where $\mathbf{p} = \mathbf{p}_a$ or $\mathbf{p}_b$. This equation can be rearranged to give:-

$$\mathbf{A}_4(\theta_4)\mathbf{A}_5(\theta_5)\mathbf{A}_6(\theta_6)\mathbf{p} = \mathbf{A}_3^{-1}(\theta_3)\mathbf{A}_2^{-1}(\theta_2)\mathbf{A}_1^{-1}(\theta_1)\mathbf{p}' \qquad (**)$$

Now, the right-hand side of the above equation is, in principle, known. Also we have chosen the points $\mathbf{p}_a$ and $\mathbf{p}_b$ to be in the same relation to the wrist centre $\mathbf{p}_c$, as the points

**a** and **b** were to the origin in section 5.3. In fact we can write:-

$$\mathbf{p}_a = \mathbf{p}_c + \mathbf{a} \qquad \text{and} \qquad \mathbf{p}_b = \mathbf{p}_c + \mathbf{b}$$

Equation $(**)$ above can now be written as:-

$$\mathbf{K}(\theta_4, \theta_5, \theta_6)\mathbf{a} + \mathbf{p}_c = \mathbf{R}(-\theta_3, \mathbf{i})\mathbf{R}(-\theta_2, \mathbf{i})\mathbf{R}(-\theta_1, \mathbf{k})\mathbf{p}'_a - \mathbf{R}(-\theta_3, \mathbf{i})\mathbf{v}$$

Here $\mathbf{K}$ is the kinematic matrix of the 3-R wrist, as in section 5.3. Since $\mathbf{p}_c$ lies on all the axes of the wrist it is not affected by the kinematics of the wrist. On the right-hand side of the equation we have the inverses of the rotations about the first three joints; the term in $\mathbf{v}$ results from the translation part of $\mathbf{A}_3$. We also get a similar equation for **b**. Now it is possible to rearrange the above equation into the form we solved in section 5.3:-

$$\mathbf{K}(\theta_4, \theta_5, \theta_6)\mathbf{a} = \boldsymbol{\alpha} \qquad \mathbf{K}(\theta_4, \theta_5, \theta_6)\mathbf{b} = \boldsymbol{\beta}$$

The vector $\boldsymbol{\alpha}$ is given by:-

$$\boldsymbol{\alpha} = \begin{pmatrix} x_a \cos\theta_1 + y_a \sin\theta_1 - D_3 \\ x_a \sin\theta_1 \cos(\theta_2+\theta_3) + y_a \cos\theta_1 \cos(\theta_2+\theta_3) + z_a \sin(\theta_2+\theta_3) - L_2 \sin\theta_3 \\ x_a \sin\theta_1 \sin(\theta_2+\theta_3) - y_a \cos\theta_1 \sin(\theta_2+\theta_3) + z_a \cos(\theta_2+\theta_3) - L_2 \cos\theta_3 - D_4 \end{pmatrix}$$

The vector $\boldsymbol{\beta}$ has a similar expression, with $x_a$, $y_a$ and $z_a$ replaced by $x_b$, $y_b$ and $z_b$. It is now just a matter of substituting these expressions into the solutions we have already found for the 3-R wrist. Although tedious, the procedure is straightforward. The result is not particularly instructive, but would be necessary for the robot's control system.

## 5.6 Inverse Kinematics of the Puma

We may summarize the results of the last two sections in the following page of equations.

$$\begin{aligned}
\cos\theta_1 &= \{D_3 x_c \pm y_c\sqrt{x_c^2 + y_c^2 - D_3^2}\}/(x_c^2 + y_c^2) \\
\sin\theta_1 &= \{D_3 - x_c \cos\theta_1\}/y_c \\
r_y &= y_c \cos\theta_1 - x_c \sin\theta_1 \\
r_z &= z_c \\
\cos\theta_3 &= \{(r_y^2 + r_z^2) - (L_2^2 + D_4^2)\}/2L_2D_4 \\
\sin\theta_3 &= \pm\sqrt{1 - \cos^2\theta_3} \\
\cos\theta_2 &= \{r_z(L_2 + D_4\cos\theta_3) - r_y D_4 \sin\theta_3\}/(r_y^2 + r_z^2) \\
\sin\theta_2 &= \{-r_y(L_2 + D_4\cos\theta_3) + r_z D_4 \sin\theta_3\}/(r_y^2 + r_z^2) \\
z_\alpha &= x_a \sin\theta_1 \sin(\theta_2+\theta_3) - y_a \cos\theta_1 \sin(\theta_2+\theta_3) + z_a \cos(\theta_2+\theta_3) \\
&\quad - L_2\cos\theta_3 - D_4 \\
\cos\theta_5 &= z_\alpha
\end{aligned}$$

$$
\begin{aligned}
\sin\theta_5 &= \pm\sqrt{1-\cos^2\theta_5} \\
x_\alpha &= x_a\cos\theta_1 + y_a\sin\theta_1 - D_3 \\
y_\alpha &= -x_a\sin\theta_1\cos(\theta_2+\theta_3) + y_a\cos\theta_1\cos(\theta_2+\theta_3) + z_a\sin(\theta_2+\theta_3) \\
&\quad - L_2\sin\theta_3 \\
\cos\theta_4 &= x_\alpha/\sin\theta_5 \\
\sin\theta_4 &= y_\alpha/\sin\theta_5 \\
z_\beta &= x_b\sin\theta_1\sin(\theta_2+\theta_3) - y_b\cos\theta_1\sin(\theta_2+\theta_3) + z_b\cos(\theta_2+\theta_3) \\
&\quad - L_2\cos\theta_3 - D_4 \\
y_\beta &= -x_b\sin\theta_1\cos(\theta_2+\theta_3) + y_b\cos\theta_1\cos(\theta_2+\theta_3) + z_b\sin(\theta_2+\theta_3) \\
&\quad - L_2\sin\theta_3 \\
\cos\theta_6 &= -z_\beta/\sin\theta_5 \\
\sin\theta_6 &= (y_\beta - \sin\theta_4\cos\theta_5\cos\theta_6)/\cos\theta_4
\end{aligned}
$$

Although this looks horribly complicated, at least a solution is possible. If we had chosen the joints arbitrarily then the tricks we used would not have worked. For such general cases analytic solutions are not possible, and usually numerical techniques have to be used. This can be a problem if the number of postures is not known; most numerical methods will only give a single solution. For the general six joint serial robot the number of postures is believed to be sixteen. How the number of postures changes as the design parameters are altered can only be guessed, at present. This is why there are so few different designs of robots: only the ones with analytic solutions for the inverse kinematics tend to be used. However, the range of designs for which the last three joint axes intersect in a common point do always have an analytic solution.

## 5.7 Parallel Manipulators

The inverse kinematics of parallel manipulators like the Stewart platform are surprisingly straightforward. In fact, it is the forward kinematics which are hard here, which is why we have not studied them earlier. To keep things simple we will only look at a planar parallel manipulator; see fig. 5.5. The mechanism has three sliding joints attached to both the ground link and the movable link by hinge joints. This means that the movable link has three degrees-of-freedom, the correct amount for a planar manipulator.

Let us choose our co-ordinates so that the hinges on the ground link are at the points $\mathbf{p}_1 = (0,0)$ and $\mathbf{p}_2 = (1,0)$. Again for convenience, let us choose the two points which determine the position and orientation of the movable link to be the centres of the hinges attached to that link. We denote them $\mathbf{a}$ and $\mathbf{b}$. Now the inverse kinematic problem is to find the joint variables given the two points $\mathbf{a}$ and $\mathbf{b}$: in this case the joint variables are the lengths of the three sliding joints. Elementary geometry gives these lengths as:-

$$d_{a1} = |\mathbf{a}-\mathbf{p}_1|, \qquad d_{b1} = |\mathbf{b}-\mathbf{p}_1|, \qquad d_{b2} = |\mathbf{b}-\mathbf{p}_2|$$

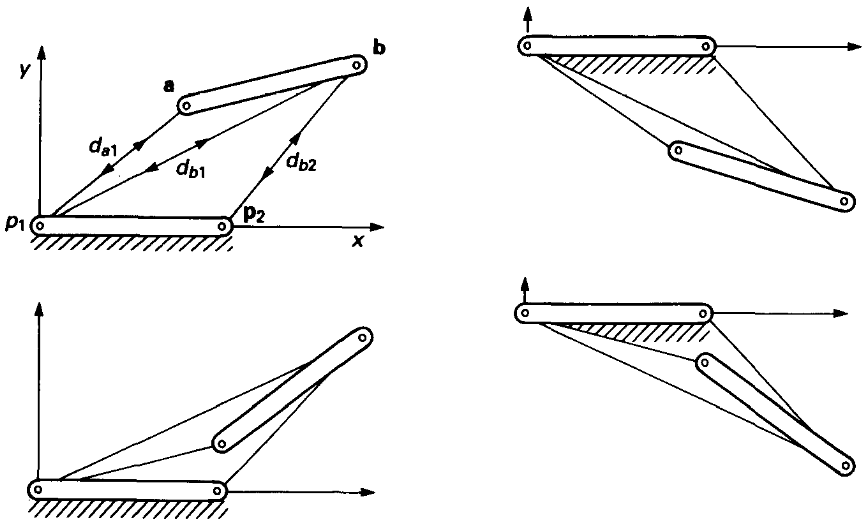


**Figure 5.5** The Planar Parallel Manipulator and its Conformations

The inverse kinematics gives a unique solution, so there are no complications with postures here.

As mentioned above, the forward kinematics is more complicated. Given the joint variables we seek the position and orientation of the movable link. From a consideration of the geometry we get four equations:-

$$
\begin{aligned}
(\mathbf{a}-\mathbf{p}_1)\cdot(\mathbf{a}-\mathbf{p}_1) &= d_{a1}^2 \\
(\mathbf{b}-\mathbf{p}_1)\cdot(\mathbf{b}-\mathbf{p}_1) &= d_{b1}^2 \\
(\mathbf{b}-\mathbf{p}_2)\cdot(\mathbf{b}-\mathbf{p}_2) &= d_{b2}^2 \\
(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= 1
\end{aligned}
$$

The middle two of these equations can be expanded to:-

$$(b_x^2 + b_y^2) = d_{b1}^2 \qquad \text{and} \qquad (b_x^2 + b_y^2) - 2b_x + 1 = d_{b2}^2$$

So we can immediately solve for $b_x$:-

$$b_x = \frac{1}{2}(1 + d_{b1}^2 - d_{b2}^2)$$

Hence we get two solutions for $b_y$:-

$$b_y = \pm\sqrt{d_{b1}^2 - b_x^2}$$

The first and last equations can now be written as:-

$$(a_x^2 + a_y^2) = d_{a1}^2 \qquad \text{and} \qquad a_x b_x + a_y b_y = d_{a1}^2 + d_{b1}^2 - 1$$

Since $b_x$ and $b_y$ are now known, the second of these equations can be considered as linear. So, assuming that $b_y \neq 0$ we can substitute for $a_y$ in the quadratic equation to give:-

$$d_{b1}^2 a_x^2 - 2(d_{a1}^2 + d_{b1}^2 - 1)b_x b_y a_x + (d_{a1}^4 + d_{b1}^4 - 3d_{a1}^2 - 2d_{b1}^2 + 2d_{a1}^2 d_{b1}^2 - 1) = 0$$

The familiar solution for a quadratic equation in one variable can now be applied:-

$$a_x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$\begin{aligned} A &= d_{b1}^2 \\ B &= -2(d_{a1}^2 + d_{b1}^2)b_x b_y \\ C &= d_{a1}^4 + d_{b1}^4 - 3d_{a1}^2 - 2d_{b1}^2 + 2d_{a1}^2 d_{b1}^2 - 1 \end{aligned}$$

Finally, we recover $a_y$ from the linear equation:-

$$a_y = \frac{1}{b_y}(d_{a1}^2 + d_{b1}^2 - 1) - a_x b_x$$

Notice that we get four possible solutions in general, called **conformations**. Different conformations have the same values for the joint variables but correspond to different positions and orientations of the end-effector. This is exactly the opposite way around to serial manipulators, but we will get the same kinds of phenomena as with a serial manipulator. The number of conformations will in general be four but will be less at singular positions.

The Stewart platform is rather harder than this example, usually the forward kinematics is done numerically. For a general manipulator, which is neither serial nor parallel, both the forward and inverse kinematics will be hard. Both will involve the solution of sets of algebraic equations and we should expect both multiple postures and conformations to be present.

## Exercises

**5.5** The wrist centre of a Puma robot is located at the following positions:-

$$\text{(i) } \mathbf{p}_c = \begin{pmatrix} 5/\sqrt{2} \\ -3/\sqrt{2} \\ -4 \end{pmatrix}, \qquad \text{(ii) } \mathbf{p}_c = \begin{pmatrix} -(2 + \frac{3}{\sqrt{2}}) \\ -(2 + \frac{5}{\sqrt{2}}) \\ \frac{4}{\sqrt{2}} \end{pmatrix}.$$

Use the inverse kinematic relations given in the previous sections to find the possible settings for the first three joint angles in each case. Take $L_2 = 4, D_3 = 1$ and $D_4 = 4$.

**5.6** Find the inverse kinematic relations for the first three joints of the Stanford manipulator, see fig 4.7. In particular find the joint variables $(\theta_1, \theta_2, d_3)$ in terms of the co-ordinates of the wrist centre $(x_c, y_c, z_c)$. How many postures does such a manipulator have?

**5.7** Consider the planar parallel manipulator introduced in section 5.7. Let its home position be when $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (1, 1)$. Suppose the movable link undergoes a rigid transformation given by the matrix:-

$$\begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Find the lengths of the sliding joint as functions of $t_x$, $t_y$ and $\theta$.