CS3500: Object-Oriented Design
Fall 2013

Class 11
10.11.2013
Plan for Today

• Debugging
• Asymptotic notation
• Efficiency
• Optimization
• Red-Black Trees
Debugging
Writing New Code

• write tests first
• write a small amount of code
  - then test it
  - repeat
Modifying Code

• write new tests
• make a small change;
  - test it;
  - repeat
Debugging Code

• Do not make random or extensive changes to the program!

• Instead, examine the code to figure out what went wrong
  - Which tests are failing?
  - Find a test that is failing - is it repeatable? (sometimes it is not)

• Think before you change anything

• Figure out why it went wrong
Checking invariants at run time.
Checking invariants at run time.

private static final boolean DEBUGGING = false;

...  

boolean contains(String s) {
    if (DEBUGGING) {
        if (!repOK()){
            ...
        }
    }
    ...
}

...
Overriding the `toString()` method when debugging
Overriding the `toString()` method when debugging

```java
@Override
public String toString () {
    if (DEBUGGING) {
        if(!repOK()){
            System.out.println("!repOK");
        }
        return toString(0);
    }
    else {
        return toString();
    }
}
```
Attitude Counts!
Asymptotic notation
Why do we care about asymptotic notation?

• Order of growth of a function

• Timing and efficiency of algorithms and implementations
Big-O
Big-O notation
[Levitin]

A function $t(x)$ is said to be in $O(g(x))$, denoted $t(x) \in O(g(x))$, if $t(x)$ is bounded above by some constant multiple of $g(x)$ for all large $x$, i.e., if there exist some positive constant $c$ and some nonnegative integer $x_0$ such that $t(x) \leq c \ast g(x)$ for all $x \geq x_0$. 
Big-O notation

[Levitin]

Big-oh notation: $t(n) \in O(g(n))$
Big-O notation

[Clinger]

More general definition:

If \( g : X \rightarrow R \), then

\[ O(g) = \{ f : X \rightarrow R | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \} \]
Is $f(x) = x \in O(x^2)$?
Is $f(x) = x \in O(x^{\wedge}2)$?

We want to find $c_0$ and $c_1 > 0$ such that

$\forall x \in X, x \leq c_0 + c_1 \times x^2$
Is \( f(x) = x \in O(x^2) \)?

We want to find \( c_0 \) and \( c_1 > 0 \) such that
\[
\forall x \in X, x \leq c_0 + c_1 \cdot x^2
\]

What about \( c_0 = 1 \) and \( c_1 = 1 \)?
Is $f(x) = x \in O(x^2)$?

We want to find $c_0$ and $c_1 > 0$ such that

$$\forall x \in X, x \leq c_0 + c_1 \cdot x^2$$

What about $c_0 = 1$ and $c_1 = 1$?

$$\forall x \in X, x \leq 1 + 1 \cdot x^2$$
Is $f(x) = x \in O(x^2)$?
Is \( f(x) = x \in O(x^2) \)?
Is \( f(x) = x \in O(x^2) \)?

If there exist some positive constant \( c \) and some nonnegative integer \( x_0 \) such that \( t(x) \leq c \times g(x) \) for all \( x \geq x_0 \)
Is \( f(x) = x \in O(x^2) \)?

If there exist some positive constant \( c \) and some nonnegative integer \( x_0 \) such that \( t(x) \leq c \cdot g(x) \) for all \( x \geq x_0 \)

We want to find: \( x \leq c \cdot x^2 \) for all \( x \geq x_0 \)
Is \( f(x) = x \in O(x^2) \)?

If there exist some positive constant \( c \) and some nonnegative integer \( x_0 \) such that \( t(x) \leq c \times g(x) \) for all \( x \geq x_0 \)

We want to find: \( x \leq c \times x^2 \) for all \( x \geq x_0 \)

What about \( c = 1 \) and \( x_0 = 1 \)?
Is $f(x) = x \in O(x^2)$?

If there exist some positive constant $c$ and some nonnegative integer $x_0$ such that $t(x) \leq c \cdot g(x)$ for all $x \geq x_0$

We want to find: $x \leq c \cdot x^2$ for all $x \geq x_0$

What about $c = 1$ and $x_0 = 1$?

$x \leq 1 \cdot x^2$ for all $x \geq 1$
Is $f(x) = x \in O(x^2)$?
Is \( f(x) = 100x + 5 \in O(x^2) \)?
Is \( f(x) = 100x + 5 \in O(x^2) \)?

\[
O(g) = \{ f : X \rightarrow \mathbb{R} \mid \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \}
\]
Is \( f(x) = 100x + 5 \in \mathcal{O}(x^2) \)?

\( \mathcal{O}(g) = \{ f : X \to \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \} \)

We want to find \( c_0 \) and \( c_1 > 0 \) such that \( \forall x \in X, 100x + 5 \leq c_0 + c_1 \cdot x^2 \).
Is \( f(x) = 100x + 5 \in O(x^2) \)?

\[
O(g) = \{ f : X \rightarrow \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \times g(x) \}
\]

We want to find \( c_0 \) and \( c_1 > 0 \) such that \( \forall x \in X, 100x + 5 \leq c_0 + c_1 \times x^2 \).

What about \( c_0 = 100 \) and \( c_1 = 100 \)?
Is \( f(x) = 100x + 5 \in O(x^2) \)?

\[ O(g) = \{ f : X \to \mathbb{R} \mid \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \} \]

We want to find \( c_0 \) and \( c_1 > 0 \) such that \( \forall x \in X, 100x + 5 \leq c_0 + c_1 \cdot x^2 \).

What about \( c_0 = 100 \) and \( c_1 = 100 \)?

\( \forall x \in X, 100x + 5 \leq 100 + 100 \cdot x^2 \)
Is $f(x) = 100x + 5 \in O(x^2)$?
Is \( f(x) = 100x + 5 \in O(x) \)?
Is \( f(x) = 100x + 5 \in O(x)? \)

\[ O(g) = \{ f : X \rightarrow \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \ast g(x) \} \]
Is \( f(x) = 100x + 5 \in O(x) \)?

\[ O(g) = \{ f : X \rightarrow \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 g(x) \} \]

We want to find \( c_0 \) and \( c_1 > 0 \) such that
\[ \forall x \in X, 100x + 5 \leq c_0 + c_1 x. \]
Is \( f(x) = 100x + 5 \in O(x) \)?

\[ O(g) = \{ f : X \rightarrow \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \} \]

We want to find \( c_0 \) and \( c_1 > 0 \) such that
\[ \forall x \in X, 100x + 5 \leq c_0 + c_1 \cdot x. \]

What about \( c_0 = 5 \) and \( c_1 = 100 \)?
Is \( f(x) = 100x + 5 \in O(x) \)?

\[ O(g) = \{ f : X \rightarrow \mathbb{R} | \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \cdot g(x) \} \]

We want to find \( c_0 \) and \( c_1 > 0 \) such that
\[ \forall x \in X, 100x + 5 \leq c_0 + c_1 \cdot x. \]

What about \( c_0 = 5 \) and \( c_1 = 100 \)?
\[ \forall x \in X, 100x + 5 \leq 100x + 5 \]
Sum of Integers from 1 to n
Sum of Integers from 1 to \( n \)

\[
\sum_{i=1}^{n} i
\]
Sum of Integers from 1 to n

\[ \sum_{i=1}^{n} i \in O(n^2) \]
∀x∈X, \(\frac{(x^2+x)}{2} \leq 1 + 5x^2\)
Theorem

If \( f(x) \) is a polynomial function of \( x \) of degree \( k \), with positive leading coefficient and restricted to non-negative \( x \), then

\[
O(f) = O(x^k).
\]
Big Omega
Big-Omega notation

[Levitin]

A function \( t(x) \) is said to be in \( \Omega(g(x)) \), denoted \( t(x) \in \Omega(g(x)) \), if \( t(x) \) is bounded below by some positive constant multiple of \( g(x) \) for all large \( x \), i.e., if there exist some positive constant \( c \) and some nonnegative integer \( x_0 \) such that \( t(x) \geq c \times g(x) \) for all \( x \geq x_0 \).
Big-Omega notation

[Levitin]
Is \( f(x) = x^3 \in \Omega(x^2) \)?
Is \( f(x) = x^3 \in \Omega(x^2) \)?

If there exist some positive constant \( c \) and some nonnegative integer \( x_0 \) such that \( t(x) \geq c \times g(x) \) for all \( x \geq x_0 \).
Is $f(x) = x^3 \in \Omega(x^2)$?

If there exist some positive constant $c$ and some nonnegative integer $x_0$ such that $t(x) \geq c \times g(x)$ for all $x \geq x_0$.

Want $x^3 \geq c \times x^2$, $\forall x \geq x_0$. 
Is $f(x) = x^3 \in \Omega(x^2)$?

If there exist some positive constant $c$ and some nonnegative integer $x_0$ such that $t(x) \geq c \cdot g(x)$ for all $x \geq x_0$.

Want $x^3 \geq c \cdot x^2$, $\forall x \geq x_0$.

What about $c = 1$ and $x_0 = 1$?
Is $f(x) = x^3 \in \Omega(x^2)$?

If there exist some positive constant $c$ and some nonnegative integer $x_0$ such that $t(x) \geq c \cdot g(x)$ for all $x \geq x_0$.

Want $x^3 \geq c \cdot x^2$, $\forall x \geq x_0$.

What about $c = 1$ and $x_0 = 1$?

$x^3 \geq 1 \cdot x^2$, $\forall x \geq 1$
Is \( f(x) = x^3 \in \Omega(x^2) \)?
Big-Theta
Big-Theta notation

[Levitin]

A function \( t(x) \) is said to be in \( \Theta(g(x)) \), denoted \( t(x) \in \Theta(g(x)) \), if \( t(x) \) is bounded both above and below by some positive constant multiples of \( g(x) \) for all large \( x \), i.e., if there exist some positive constant \( c_1 \) and \( c_2 \) and some nonnegative integer \( x_0 \) such that \( c_2 \times g(x) \leq t(x) \leq c_1 \times g(x) \) for all \( x \geq x_0 \).
Big-Theta notation

[Levitin]

Big-theta notation: $t(n) \in \Theta(g(n))$
Is $f(x) = x^3 + 4 \cdot x^2 + 5 \in \Theta(x^3)$?
Is \( f(x) = x^3 + 4 \cdot x^2 + 5 \in \Theta(x^3) \)?

If there exist some positive constant \( c_1 \) and \( c_2 \) and some nonnegative integer \( x_0 \) such that
\[
c_2 \cdot g(x) \leq t(x) \leq c_1 \cdot g(x)
\]
for all \( x \geq x_0 \).
Is \( f(x) = x^3 + 4 \cdot x^2 + 5 \in \Theta(x^3) \)?

If there exist some positive constant \( c_1 \) and \( c_2 \) and some nonnegative integer \( x_0 \) such that
\[
c_2 \cdot g(x) \leq t(x) \leq c_1 \cdot g(x) \text{ for all } x \geq x_0.
\]

Want \( c_2 \cdot x^3 \leq x^3 + 4 \cdot x^2 + 5 \leq c_1 \cdot x^3, \forall x \geq x_0. \)
Is \( f(x) = x^3 + 4 \times x^2 + 5 \in \Theta(x^3) \)?

If there exist some positive constant \( c_1 \) and \( c_2 \) and some nonnegative integer \( x_0 \) such that

\[
c_2 \times g(x) \leq t(x) \leq c_1 \times g(x) \quad \text{for all} \quad x \geq x_0.
\]

Want

\[
c_2 \times x^3 \leq x^3 + 4 \times x^2 + 5 \leq c_1 \times x^3,
\]

\( \forall x \geq x_0. \)

What about \( c_1 = 10, c_2 = 1 \) and \( x_0 = 1 \)?
Is \( f(x) = x^3 + 4 \times x^2 + 5 \in \Theta(x^3) \)?

If there exist some positive constant \( c_1 \) and \( c_2 \) and some nonnegative integer \( x_0 \) such that

\[
c_2 \times g(x) \leq t(x) \leq c_1 \times g(x) \text{ for all } x \geq x_0.
\]

Want \( c_2 \times x^3 \leq x^3 + 4 \times x^2 + 5 \leq c_1 \times x^3, \forall x \geq x_0. \)

What about \( c_1 = 10, c_2 = 1 \) and \( x_0 = 1 \)?

\[
1 \times x^3 \leq x^3 + 4 \times x^2 + 5 \leq 10 \times x^3, \forall x \geq 1
\]
Is $f(x) = x^3 + 4 \cdot x^2 + 5 \in \Theta(x^3)$?
Let’s play a game...
Let’s play a game...

\[ f(x) = x \times (x + 1)/2 \]
Let’s play a game...

\[ f(x) = x \times (x + 1)/2 \]

Let’s assume \( f \in O(1) \), then (by the general definition of big-O) there exist constants \( c_0 \) and \( c_1 \) such that

\[ \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \times 1 = c_0 + c_1 \]
Let’s play a game...

\[ f(x) = x \times (x + 1)/2 \]

Let’s assume \( f \in O(1) \), then (by the general definition of big-O) there exist constants \( c_0 \) and \( c_1 \) such that

\[ \exists c_0, c_1 > 0, \forall x \in X, f(x) \leq c_0 + c_1 \times 1 = c_0 + c_1 \]

Pick values for \( c_0 \) and \( c_1 \).
Efficiency
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<th>( \text{log}_2 n )</th>
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Efficiency

• Worst-case efficiency
• Best-case efficiency
• Average-case efficiency
Efficiency of Assignments with Recipe Implementations

- FSetString
- FListInteger
Binary Search Tree Efficiency
\[
\left(1 \times 1 + 2 \times 2 + 4 \times 3\right)/7 = 17/7 \approx 2.42
\]
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Optimization
“More computing sins are committed in the name of efficiency (without necessarily achieving it) than for any other single reason—including blind stupidity.”

–W.A. Wulf
"We should forget about small efficiencies, say about 97% of the time: premature optimization is the root of all evil"

–Donald Ervin Knuth
Rules of Optimization

1. Don't.

2. Don't yet.

3. Don't optimize more than necessary.
Red-Black Trees
Pop Quiz on Reading
“Red-Black Trees in a Functional Setting”

1. What are the two balance invariants for red-black trees?

2. Why are red-black trees used?
Red-Black Trees

Red-black trees are binary search trees in which each node has a color (either red or black) and the following balancing invariants are preserved by all operations:

1. No red node has a red child.
2. Every path from the root to an empty tree/node contains the same number of black nodes.