CS3000

5/11 - Thurs :)

Admin

- no rec today (first thurs fin one is 5/18)
- Short HW1 due 9pm
- Long ~~[illegible]~~ HW1 out, due Tues. 9pm

Agenda

1. Divide and conquer on a number
2. Karatsuba algorithm
3. Algorithm Analysis

$\boxed{\text{Math Recap}}$

ex $\displaystyle\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2} = \Theta(n^2)$

ex $\displaystyle\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1} = \Theta(x^n)$

ex $a^{\lg c} = c^{\lg a}$ $\quad c=8 \quad a=4$

$4^{\lg 8} = 4^3 = 8^{\lg 4} = 8^2 = 64$

ex $2^{\lg n} = n \qquad 2^{\lg n - 1} = \frac{n}{2}$

I. Divide + Conquer on a number

Linear Search → Bin Search

$\Theta(n)$ $\Theta(\lg n)$ W.C.

↳ divide + conquer on BST

Today: input is an integer with n digits

→ divide

$n/2$ | $n/2$

Using a number with n digits in $\boxed{\text{multiplication}}$

ex Addition first

$n = 4$

$w = 1234$

$y = 1122$

$w + y$

$$\begin{array}{r} 1\;2\;3\;4 \\ +\;1\;1\;2\;2 \\ \hline 2\;3\;5\;6 \end{array}$$

$\Theta(n)$ algorithm

ex Multiplication

$w = 1234$

$y = 1122$

$w \times y$

$$\begin{array}{r} 1\;2\;3\;4 \\ \times\;1\;1\;2\;2 \\ \hline 2\;4\;6\;8 \\ 2\;4\;6\;8\;0 \end{array}$$

$\Theta(n^2)$

$$\begin{array}{rrrrrrr}
 & 1 & 2 & 3 & 4 & 0 & 0 \\
1 & 2 & 3 & 4 & 0 & 0 & 0 \\
\hline
1 & 3 & 8 & 4 & 5 & 4 & 8
\end{array}$$

"Can we do better?"

↳ maybe with a known ~~[illegible]~~ algorithmic technique!

$\boxed{\text{Insights}}$

- adding on zeros is easy
  - ↳ multiply by a power of 10 does not need multiplication
- express a decimal number with powers of 10
  - ↳ $1234 = 1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10^1 + 4 \cdot 10^0$
  - ↳ $1234 = 12 \cdot 10^2 + 34 \cdot 10^0$
    $= 1200 + 34$
    $= 1234$

$w = 1234 = 12 \cdot 10^2 + 34 \cdot 10^0$

$y = 1122 = 11 \cdot 10^2 + 22 \cdot 10^0$

→ want to keep mults to $n/2$ digits

↓

$W \times Y = 1234 \times 1122$

$= (12 \cdot 10^2 + 34)(11 \cdot 10^2 + 22)$

$= 10^4 \cdot 12 \cdot 11 + 34 \cdot 11 \cdot 10^2 + 22 \cdot 12 \cdot 10^2 + 34 \cdot 22$

$= 10^4 \cdot \underline{12 \cdot 11} + 10^2(\underline{34 \cdot 11} + \underline{22 \cdot 12}) + \underline{34 \cdot 22}$

$n/2$ mult. $\quad$ $n/2$ mult

$= 1384548$

!! ☺

? Actually faster?

2. Karatsuba Algorithm

$n = 4$

$W = \underbrace{1\ 2}_{a}\ \underbrace{3\ 4}_{b}$ $\qquad$ $Y = \underbrace{11}_{c}\ \underbrace{22}_{d}$

$W = 10^{n/2} \cdot a + b$ $\qquad$ $Y = 10^{n/2} \cdot c + d$

$W \times Y = (10^{n/2} \cdot a + b)(10^{n/2} \cdot c + d)$

$= 10^n \cdot a \cdot c + 10^{n/2} \cdot b \cdot c + 10^{n/2} \cdot a \cdot d + b \cdot d$

$= 10^n \cdot a \cdot c + 10^{n/2}(b \cdot c + a \cdot d) + b \cdot d$

!! ☺

The Algo

KAR(w, y, n)
if n == 1
  return w · y
m = ⌈n/2⌉
// Split w into a, b and y into c, d
e = KAR(a, c, m)   a · c   → divide
f = KAR(b, d, m)   b · d
g = KAR(b−a, c−d, m)   (b−a)(c−d)
return $10^{2m} e + 10^m (e + f + g) + f$ → combine

→ conquer

ex w = 48   n = 2
  y = 34

48 · 34 = ? 1632 !!

↳ m = 1
  Split w = 4 8   y = 3 4
       a b   c d

e = KAR(4, 3, 1) → 12
f = KAR(8, 4, 1) → 32
g = KAR(4, −1, 1) → −4

return $10^2 \cdot 12 + 10(12 + 32 - 4) + 32$

↳ 1200 + 400 + 32
↳ 1632

10:47

3. Algo Analysis

- Run Time
- Correctness

$\boxed{T(n) = \ldots ??}$

KAR (w, y, n)

if n == 1
    return w · y

Base Case $T(1) = 2$

m = ⌈n/2⌉

// Split w into a, b and y into c, d

c

e = KAR(a, c, m)

f = KAR(b, d, m)

g = KAR(b−a, c−d, m)

$3 \cdot T(n/2)$

return $10^{2m} e + 10^m (e + f + g) + f$

$c \cdot n$

$T(n) = 3 \cdot T(n/2) + cn + c$

$T(1) = 2$

run time. need to solve it!

Iteration method

1st $T(n) = 3 \cdot T(n/2) + cn + c$

Plugin: $T(n/2) = 3 \cdot T(n/4) + c\frac{n}{2} + c$

2nd $T(n) = 3\left[3 \cdot T(n/4) + \frac{1}{2}cn + c\right] + cn + c$

$= 9\,T(n/4) + \frac{3}{2}cn + cn + 3c + c$

Plug in: $T(n/4) = 3 \cdot T(n/8) + c \cdot \frac{n}{4} + c$

3rd $\quad T(n) = 9\left[3 \cdot T(n/8) + \frac{1}{4}cn + c\right] + \frac{3}{2}cn + cn + 3c + c$

$= 27\,T(n/8) + \underbrace{\frac{9}{4}cn + \frac{3}{2}cn + cn}_{\frac{3^2}{2^2}cn \quad \frac{3^1}{2^1}cn \quad \frac{3^0}{2^0}cn} + \underbrace{9c + 3c + c}_{3^2c \quad 3^1c \quad 3^0c}$

kth

$$\sum_{i=0}^{k-1}\left(\frac{3}{2}\right)^i \cdot cn \qquad\qquad \sum_{i=0}^{k-1} 3^i \cdot c$$

$$cn\sum_{i=0}^{k-1}\left(\frac{3}{2}\right)^i \qquad\qquad c \cdot \sum_{i=0}^{k-1} 3^i$$

$$cn \cdot \Theta\left(\frac{3}{2}\right)^{k-1} \qquad\qquad c \cdot \Theta(3)^{k-1}$$

→ Bounds on sums

kth

$$\boxed{T(n) = 3^k \cdot T(n/2^k) + cn\left(\frac{3}{2}\right)^{k-1} + c \cdot (3)^{k-1}}$$

Choose a value for k to get to base case $T(1) = 2$

$k = \lg n$

$$T(n) = 3^{\lg n} \cdot T(n/n) + cn\left(\frac{3^{\lg n - 1}}{2^{\lg n - 1}}\right) + c \cdot 3^{\lg n - 1}$$

$$= 3^{\lg n} \cdot 2 + c\cancel{n} \cdot \frac{2}{\cancel{n}} \cdot \left(3^{\lg n - 1}\right) + c \cdot 3^{\lg n - 1}$$

$$= 3^{\lg n} \cdot 2 + 2c \cdot (3^{\lg n - 1}) + c \cdot 3^{\lg n - 1}$$

$$= \Theta(3^{\lg n})$$

$$= \Theta(n^{\lg 3}) = \Theta(n^{1.59})$$

$\rightarrow$ Beats $\Theta(n^2)$!

$\boxed{\text{Proof of Correctness}}$ $\rightarrow$ by induction on n
(n is # digits)

$$\boxed{w \cdot y = 10^n \cdot a \cdot c + 10^{n/2}(b \cdot c + a \cdot d) + b \cdot d}$$

Base Case: $n = 1$

↳ KAR returns $w \cdot y$. done!

IH KAR works correctly when w, y have
1 digit, 2 digits, ..., $k-1$ digits
↳ strong ind.

Want to show it works with k digits

In algo...

$m = k/2$

$e = KAR(a, c, m)$
$f = KAR(d, b, m)$
$g = KAR(b-a, c-d, m)$

$\Big\}$ these work! By ind. Hyp!

return $10^{2m} \cdot e + 10^m (e + f + g) + f$

$10^k \cdot (a \cdot c) + 10^{k/2}(ac + bd + (b-a)(c-d)) + bd$

$= 10^k ac + 10^{k/2}(ac + bd + bc - ac - bd + ad) + bd$

$$\boxed{= 10^k ac + 10^{k/2}(bc + ad) + bd}$$

✓ done!

:(

- Algo's run time is $\Theta(n^{1.59})$

  is better! $\Theta(n^2)$

- Algo's correctness with induction