CS3000: Algorithms & Data — Summer 2025 — Laney Strange

Recitation 2
Date: May 13th, 2025

Name: Sample Solution

- Recitation problems are for practice only. We'll go over the solutions during your scheduled recitation on Tuesday!
- We will provide `.tex` starter files for recitations, just as we do for homeworks. For most recitations, we encourage you to work out your solution in LaTeXto practice with typesetting.
- Collaboration is strongly encouraged during recitation!

Problem 1. *Proof by Induction - Summation*

Prove that the sum of the first n positive integers is $\frac{(n)(n+1)}{2}$

Predicate $S(n)$ states that $\sum_{i=1}^{n} i = \frac{(n)(n+1)}{2}$.

Logic Statement $\forall n \in \mathbb{Z}^+, S(n)$

In your solution below, make sure you include the base case, inductive hypothesis (assume true for $S(k)$ for an arbitrary k), and inductive step (show $S(k) \implies S(k+1)$).

Solution:

Base Case $S(1)$ by definition gives us $\sum_{i=1}^{1} i = 1$. The formula gives us $\frac{1 \cdot 2}{2} = 1$. Done!

Inductive Step Let us show that $S(k) \implies S(k+1)$ for an arbitrary k.

Our **Inductive Hypothesis** is that $S(k)$ holds, i.e., $\sum_{i=1}^{k} i = \frac{(k)(k+1)}{2}$

Solution Option 1

Our intention is to show $S(k+1)$ i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

We now have, by definition of summation:

$$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + (k+1)$$

Pull out the final term:

$$= \sum_{i=1}^{k} i + (k+1)$$

By inductive hypothesis:

$$= \frac{(k)(k+1)}{2} + (k+1)$$

Multiply second term by 2/2:

$$= \frac{(k)(k+1)}{2} + \frac{2 \cdot (k+1)}{2}$$

Add numerators:

$$= \frac{(k)(k+1) + 2 \cdot (k+1)}{2}$$

Distributive law:

$$= \frac{(k+1)(k+2)}{2}$$

Done!

Solution Option 2

Our intention is to show $S(k+1)$ i.e., $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

We now have, by definition of summation:

$$\sum_{i=1}^{k+1} i = 1 + 2 + 3 + \ldots + k + (k+1)$$

Pull out the final term:

$$= \sum_{i=1}^{k} i + (k+1)$$

By inductive hypothesis:

$$= \frac{(k)(k+1)}{2} + (k+1)$$

Multiply second term by 2/2:

$$= \frac{(k)(k+1)}{2} + \frac{2 \cdot (k+1)}{2}$$

Add numerators:

$$= \frac{(k)(k+1) + 2 \cdot (k+1)}{2}$$

Multiply terms:

$$= \frac{k^2 + k + 2k + 2}{2}$$

Add terms:

$$= \frac{k^2 + 3k + 2}{2}$$

Polynomial:

$$= \frac{(k+1)(k+2)}{2}$$

Done!

Problem 2. *Proof by Induction - Correctness*

Consider the pseudocode below for a recursive algorithm.

```
RECURSIVE(n)
1  if n == 1
2      return 1
3  return n + RECURSIVE(n − 1)
```

- What would this function return in the following examples?

  1. RECURSIVE(1)

     Solution:
     1, this is the base case

  2. RECURSIVE(2)

     Solution:
     This returns $2 + \text{RECURSIVE}(1) = 2 + 1 = 3$

  3. RECURSIVE(3)

     Solution:
     This returns $3 + \text{RECURSIVE}(2) = 3 + 2 + \text{RECURSIVE}(1) = 3 + 2 + 1 = 6$

  4. RECURSIVE(4)

     Solution:
     This returns $4 + \text{RECURSIVE}(3) = 4 + 3 + \text{RECURSIVE}(2) = 4 + 3 + 2 + \text{RECURSIVE}(1) = 4 + 3 + 2 + 1 = 10$

- Show by mathematical induction that, in general, RECURSIVE(n) returns $n+(n-1)+(n-2)+...+1$ for all positive integers n.

  Solution:

  Statement $S(n)$ states that RECURSIVE(n) returns $n + (n-1) + (n-2) + ... + 1$

  Claim $\forall n \in \mathbb{Z}^+ S(n)$

  Base Case Proven in first part

  Inductive Step We will show that $S(k) \implies s(k+1)$.

  Inductive Hypothesis Assume $S(k)$, i.e., RECURSIVE(k) returns $k + (k-1) + (k-2) + ... + 1$.

  We now turn to RECURSIVE($k+1$). Because we are not in the base case, we execute line 3. We add $k+1$ to the result of RECURSIVE(k). By the inductive hypothesis, RECURSIVE(k) $= k + (k-1) + ... + 1$, so in total we have $(k+1) + k + (k-1) + ... + 1$. Done!