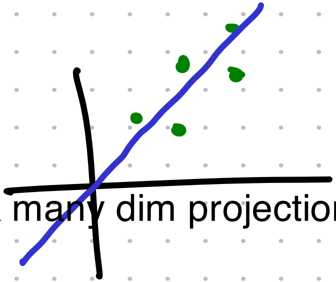
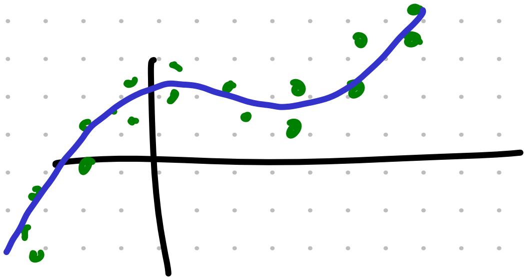
CS 2810 Day 8
Feb 11 2022

Computing with python / numpy
- build matrices, add/scale/multiply matrices, load matrix from csv, get inverse
get a row/col of matrix

Line of best fit (1d) & 1dim projections



Polynomial of best fit (many dimensions) & many dim projections



Computing with python / numpy
- build matrices
- add/scale/multiply matrices
- get inverse of a matrix:
    (whats an inverse?) A^{-1}
- get a row/col of matrix

- load matrix from csv:
    np.genfromtxt('mystery_matrix.csv', delimiter=',')

ICA 1:
You are welcome to handwrite / screenshot the answers to each question below, without showing the code used, for your ICA submission.

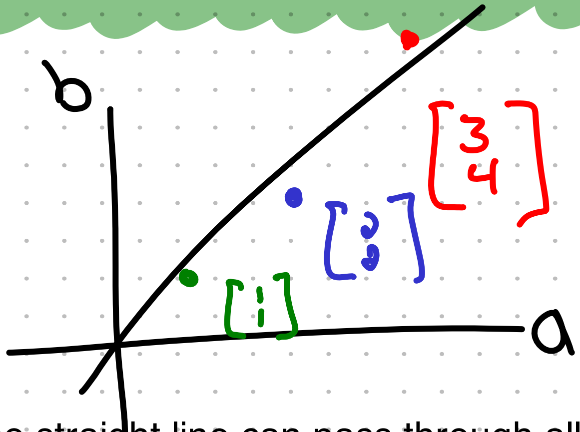
- Compute the matrix multiplication given below:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 10 & 11 \\ 12 & 13 \\ 14 & 15 \end{bmatrix}$$

- Load the data given in 'mystery_matrix.csv' given on website:

- What is the dot product of the second and fourth columns of the mystery matrix above?

LINE OF BEST FIT (1 DIM)



Goal: Find a line of the form: b = ma which best fits data

$$\vec{b} = m\vec{a}$$

SLOPE

$$1 = m\,1$$
$$2 = m\,2$$
$$4 = m\,3$$

no straight line can pass through all 3 points
-> no m exists with b = ma

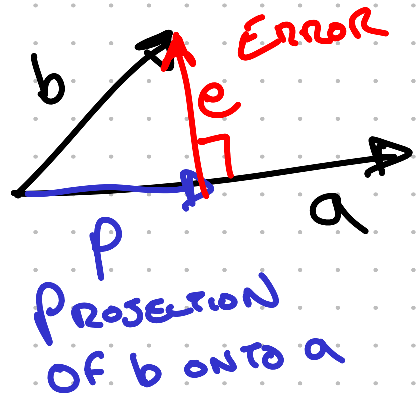
instead, what if we find the vector p which
- has a solution p = ma
    - p is in the span of {a}
- is closest to b (llb-pll is minimized)

$$p = ma$$

$$\underset{\vec{b}}{\begin{bmatrix}1\\2\\4\end{bmatrix}} = m\underset{\vec{a}}{\begin{bmatrix}1\\2\\3\end{bmatrix}}$$

1 dimensional projections (often just called projections)

Among all vectors in the span of {a}, which is closest to b?



Some observations:

obs1: e is at a right angle (orthogonal) to a

obs2: b = p + e

obs3: p is in the span of a

SOME SCALAR

③ $p = ca$

② $e = b - p$

① $e^T a = 0 \rightarrow (b-p)^T a = 0 \rightarrow (b - ca)^T a = 0$

$$(b - ca)^T a = 0$$

$$\left(b^T - ca^T\right) a = 0$$

$$b^T a - c a^T a = 0$$

$$b^T a = c\, a^T a$$

$$c = \frac{b^T a}{a^T a}$$

$$ca = p = \frac{b^T a}{a^T a} a$$

$e^T a = 0$

$$\frac{e \cdot a}{\|e\|\|a\|} = \cos 90 = 0$$

$e \cdot a = 0$

$$e^T a = \begin{array}{cc} (1\times 2) & (2\times 1) \\ \begin{bmatrix} e_0 & e_1 \end{bmatrix} & \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \end{array} = e_0 a_0 + e_1 a_1 = e \cdot a$$

$$p = \frac{b^T a}{a^T a} a = \frac{b \cdot a}{a \cdot a} a = \frac{b \cdot a}{\|a\|^2} a$$

$$\underset{(2\times 1)}{\begin{bmatrix} e_0 \\ e_1 \end{bmatrix}} \underset{(1\times 2)}{\begin{bmatrix} a_0 & a_1 \end{bmatrix}} = \begin{bmatrix} e_0 a_0 & e_0 a_1 \\ e_1 a_0 & e_1 a_1 \end{bmatrix}$$

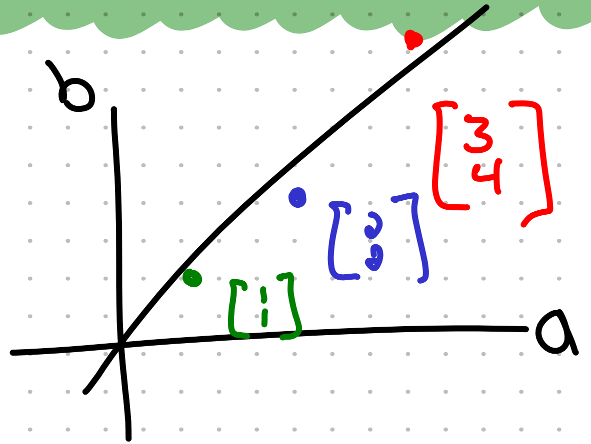
$$0 = (b - ac)^T a = b^T a - c\, a^T a$$

$$b^T a = c\, a^T a$$

$$c = \frac{b^T a}{a^T a}$$

③ $p = ca = \boxed{\frac{b^T a}{a^T a} a = p}$

Line of Best Fit (1 Dim)



Goal: Find a line of the form: b = ma which best fits data

$$b = m a$$

No solutions

$$\begin{bmatrix}1\\2\\4\end{bmatrix} = m\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Solve $p = ma$ instead

$$p = m a$$

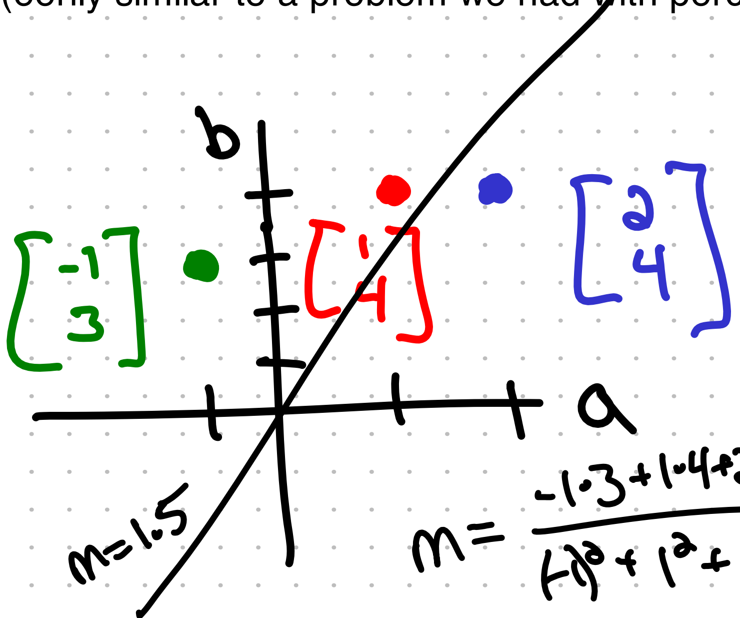
$$\frac{17}{14}\begin{bmatrix}1\\2\\3\end{bmatrix} = m\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Project b onto a

$$p = \frac{b^T a}{a^T a} a = \frac{1\cdot 1 + 2\cdot 2 + 4\cdot 3}{1\cdot 1 + 2\cdot 2 + 3\cdot 3}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{17}{14}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

ICA 2:

1. Find the projection of $\overset{b=}{[2, 3, 4]^T}$ in the span of $\overset{a=}{[1, 2, 3]^T}$
2. Using a projection, find the line of best fit of the form b = ma for the scatter points shown below
3. There's a glaring problem with our line of best fit above, what is it? How might you fix it? (eerily similar to a problem we had with perceptrons having to pass through the origin too...)



$$\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} m = \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}$$

$$a \quad m = b$$

$$am = \frac{a \cdot b}{a \cdot a} a$$

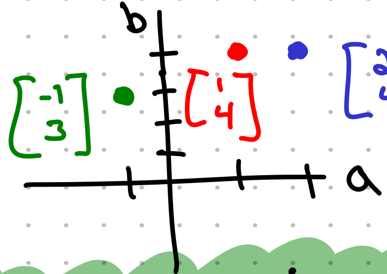
$$m = \frac{-1 \cdot 3 + 1 \cdot 4 + 2 \cdot 4}{(-1)^2 + 1^2 + 2^2} = \frac{9}{6}$$

$$p = \frac{a \cdot b}{a \cdot a} a = \frac{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4}{1^2 + 2^2 + 3^2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \frac{20}{14} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Line of Best Fit (Many Dimensions)



Old Model $b = ma$

$3 = -1m$

$4 = 1m$

$4 = 2m$

$$\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = m \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

New Model

$b = m_0 + m_1 a$

$3 = m_0 + -1 m_1$

$4 = m_0 + 1 m_1$

$4 = m_0 + 2 m_1$

$$\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = m_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + m_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}}_{b} = m_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + m_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}}_{m}$$

b = Am has no solutions ...

what if we solve p = Am instead where p is

- in the span of the columns of A
  - (there is a linear combination of cols of A which equals p ... it's our target vector m!)
- closest to b

... we need a way of projecting a vector into the span of many vectors (previously only one)

Polynomial of Best Fit (Many Dimensions)

Newer Model

$$b = m_0 + m_1 a + m_2 a^2$$

$$3 = (-1)^0 m_0 + -1^1 m_1 + m_2 (-1)^2$$

$$4 = (1)^0 m_0 + 1^1 m_1 + m_2 (1)^2$$

$$4 = (2)^0 m_0 + 2^1 m_1 + m_2 (2)^2$$

$$\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} = m_0 \begin{bmatrix} -1^0 \\ 1^0 \\ 2^0 \end{bmatrix} + m_1 \begin{bmatrix} -1^1 \\ 1^1 \\ 2^1 \end{bmatrix} + m_2 \begin{bmatrix} -1^2 \\ 1^2 \\ 2^2 \end{bmatrix}$$

New Model

$$b = m_0 + m_1 a$$

$$3 = m_0 + -1 m_1$$

$$4 = m_0 + 1 m_1$$

$$4 = m_0 + 2 m_1$$

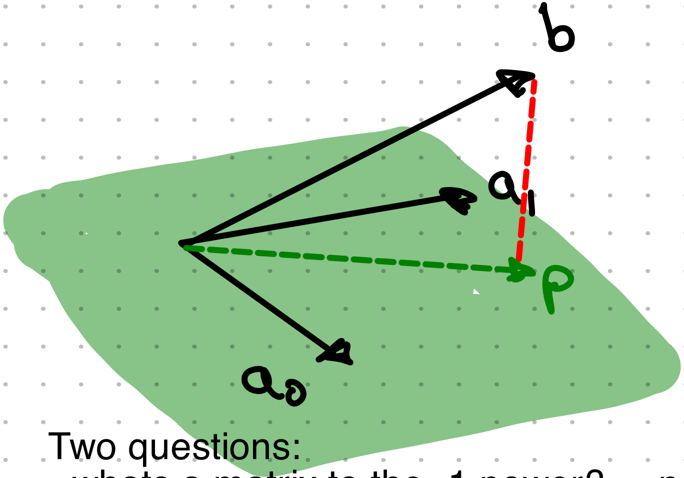
$$\underbrace{\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}}_{b} = m_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + m_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m_0 \\ m_1 \end{bmatrix}}_{m}$$

$$\underbrace{\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}}_{b} = m_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + m_1 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + m_2 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m_0 \\ m_1 \\ m_2 \end{bmatrix}}_{m}$$

$$b = Am$$

PROJECTIONS (MULTI DIMENSIONAL)

Goal: find the vector p, in the span of {a_0, a_1, ... } which is closest to b

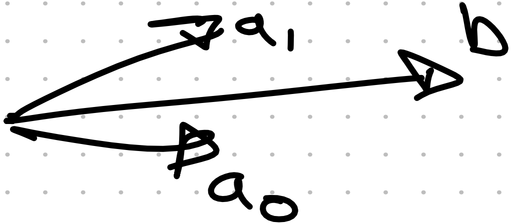


WITHOUT EXPLANATION

$$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$$

$$p = A(A^TA)^{-1}A^Tb$$

Two questions:
- whats a matrix to the -1 power? ... next slide
- do you expect students to compute that mess by hand?



$$b = Am$$

$$p = A(A^TA)^{-1}A^Tb$$

$$= A(A^TA)^{-1}(A^TA)m$$

$$= Am = b$$

SINCE $Am = p = A(A^TA)^{-1}A^Tb$

$$m = (A^TA)^{-1}A^Tb$$

COEFFICIENTS OF LINE OF BEST FIT

ICA3:
Load the matrices A and b from A.csv and b.csv respectively. (See zip next to today's notes on site)

A is the matrix with the polynomial manipulation shown here applied:
- column 0 is a to the 0th power
    - also known as: bias term
- column 1 is a to the 1st power
- column 2 is a to the 2nd power

b is a vector which contains

- where are the scatter points within A and b we're building the line of best fit to?
- Using numpy and python, find the m vector which defines the polynomial of best fit between a and b
- What polynomial is represented by this particular m vector? (Please round to the nearest integer)