CS2810 Day 6 Feb 4

→ Matrix-Matrix Multiplication

→ Linear Combinations

→ Matrix Multiplication as a Function (Transformation)

→ Building Matrix Function From Linearity

→ Scaling

→ Rotating

→ Composing Matrix Functions $BAx = y$

Matrix-Matrix Multiplication: Shape Rule

N rows

$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}$ has shape 3×2 N col

Let A has shape $n_a \times m_a$
B has shape $n_b \times m_b$

Matrix multiplication only defined when $m_a = n_b$

AB, when defined, has shape $n_a \times m_b$

SHAPE RULE EXAMPLE

A HAS SHAPE 10×17

B HAS SHAPE 17×14

GIVE OUTPUT SHAPE OF EACH OPERATION

AB

$(10 \times 17)(17 \times 14)$

(10×14)

BA

$(17 \times 14)(10 \times 17)$

UNDEFINED

TRANSPOSE OF A MATRIX SWAPS ROWS/COLS

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2×3 $\quad$ 3×2

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

3×1 $\quad$ 1×3

ICA 1 Give shape of each matrix product if it exists

$$\text{Shape}(A) = 3\times 3 \qquad \text{Shape}(B) = 3\times 1$$

$$\text{Shape}(C) = 3\times 4 \qquad \text{Shape}(D) = 1\times 4$$

AB

$(3\times 3)(3\times 1)$

(3×1)

CA

$(3\times 4)(3\times 3)$

Not defined

AC

$(3\times 3)(3\times 4)$

(3×4)

DC^TB

$(1\times 4)(4\times 3)(3\times 1)$

$(1\times 3)(3\times 1)$

(1×1)

Matrix-Matrix Multiplication: Computing (Dot Product)

$$\underset{(3\times 2)}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 7 \end{bmatrix}} \underset{(2\times 2)}{\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}} = \underset{(3\times 2)}{\begin{bmatrix} 3 & \sim \\ \sim & 3 \\ -8 & \sim \end{bmatrix}}$$

$1 \cdot 3 + 0 \cdot -2 = 3$

$1 \cdot -1 + 1 \cdot 4 = 3$

$2 \cdot 3 + 7 \cdot -2 = -8$

Each element in product matrix is dot product of corresponding row (left matrix) and col (right matrix)

LINEAR COMBINATION

(WEIGHTED SUM)

COULD BE ANY OBJECTS, MOST OFTEN MATRICES FOR US

A LINEAR COMBINATION OF $\overbrace{X_0, X_1, X_2}$, ...

IS $\alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2 + \ldots$

WHERE EACH α_i ARE SCALARS

MATRIX - VECTOR MULTIPLICATION

(IN THAT ORDER)

$$\underbrace{A}_{}\,x = \underbrace{\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x} = \begin{bmatrix} 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \\ 1\cdot 7 + 2\cdot 8 + 3\cdot 9 \\ 1\cdot 10 + 2\cdot 11 + 3\cdot 12 \end{bmatrix}$$

MATRIX-VECTOR MULTIPLICATION

(IN THAT ORDER)

$$Ax = \underbrace{\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} 1\cdot 4 + 2\cdot 5 + 3\cdot 6 \\ 1\cdot 7 + 2\cdot 8 + 3\cdot 9 \\ 1\cdot 10 + 2\cdot 11 + 3\cdot 12 \end{bmatrix}$$

Matrix vector multiplication
is a linear combination
of the columns of the matrix

$$= 1\cdot \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$$

ICA 1.5

BUILD THE MATRIX A WHICH, WHEN MULTIPLIED AS Ax, ALWAYS YIELDS THE SAME VECTOR x

$$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = A \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

WHAT IS A?

$$= x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

VECTOR-MATRIX MULTIPLICATION

IN THAT ORDER

$$xA = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 1\cdot 4 + 2\cdot 7 + 3\cdot 10, & 1\cdot 5 + 2\cdot 8 + 3\cdot 11, & 1\cdot 6 + 2\cdot 9 + 3\cdot 12 \end{bmatrix}$$

$(1,3) \quad (3,3) \quad\quad (1,3)$

$$= 1\cdot\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} + 2\cdot\begin{bmatrix} 7 & 8 & 9 \end{bmatrix} + 3\cdot\begin{bmatrix} 10 & 11 & 12 \end{bmatrix}$$

xA

VECTOR-MATRIX ~~MULTIPLICATION~~ IS A LINEAR COMBO OF ROWS OF A

TRANSPOSE IDENTITIES AND HOW THEY RELATE MATRIX-VECTOR TO VECTOR-MATRIX MULTIPLICATIONS

$$(A^T)^T = A \qquad (xA)^T = A^T x^T$$

$$(AB)^T = B^T A^T \qquad (Ax)^T = x^T A^T$$

MATRIX-VECTOR AND VECTOR MATRIX MULTIPLICATIONS ARE A TRANSPOSE AWAY FROM EACH OTHER

CONVENTION:

PREFER MATRIX-VECTOR MULTIPLICATION WHERE POSSIBLE (COL VECTORS)

(AVOID VECTOR-MATRIX W/ ROW VECTORS)

IN GENERAL, IT IS NOT ALWAYS TRUE

THAT $AB = BA$

MATRIX MULTIPLICATION IS
NOT COMMUTATIVE

ICA 2   Simplify Expressions Below

$$\begin{bmatrix} 1 & 0 & -4 \\ 1 & 0 & -4 \\ 1 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} -11 \\ -11 \\ -11 \end{bmatrix} = -11 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(3×3)

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$1/3 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1/3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1/3 \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

1×3 $\quad$ 3×1 $\quad\quad$ 1×1

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3$$

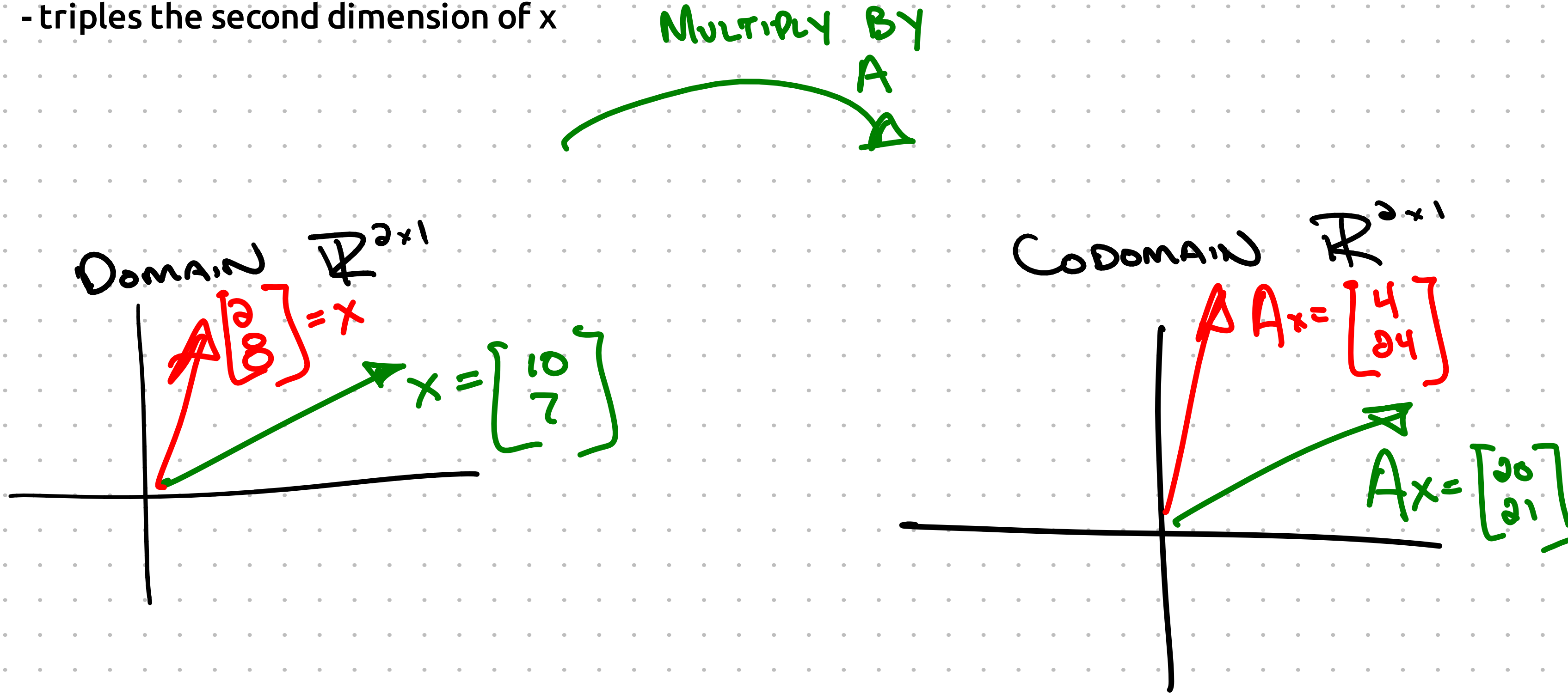
Matrix-Vector Multiplication as a Function

Let $A \in \mathbb{R}^{2\times 2}$ consider $f: \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$

$$f(x) = Ax = b$$

Given a 2d column vector x, find the matrix A which, when multiplied as Ax, yields an output which:

- doubles the first dimension of x
- triples the second dimension of x



$$Ax = b$$

$$(2\times\cancel{2})\;(\cancel{2}\times 1)\qquad(2\times 1)$$

$$(2\times 1)$$

BUILDING TRANSFORM A

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

LET a_0, a_1 BE COLUMNS OF $A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$

LET $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ WANT A WITH

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} | \\ a_0 \\ | \end{bmatrix} + 0 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}$$

LET $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ WANT A WITH

$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} | \\ a_0 \\ | \end{bmatrix} + 1 \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}$$

BUT YOU JUST BUILT $\underline{A}$ FROM ONLY TWO INPUT VECTORS, HOW DO YOU KNOW IT WORKS FOR OTHERS?

MATRIX MULTIPLICATION (BY A) IS LINEAR

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

BUT YOU JUST BUILT A FROM ONLY TWO INPUT VECTORS, HOW DO YOU KNOW IT WORKS FOR OTHERS?

$X = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ EXPECT $Ax = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$

$$A\begin{bmatrix} 7 \\ -2 \end{bmatrix} = A\left(7 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 7A\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 7\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$$

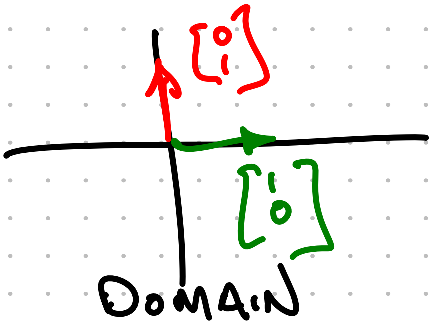
$$\begin{bmatrix} 14 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

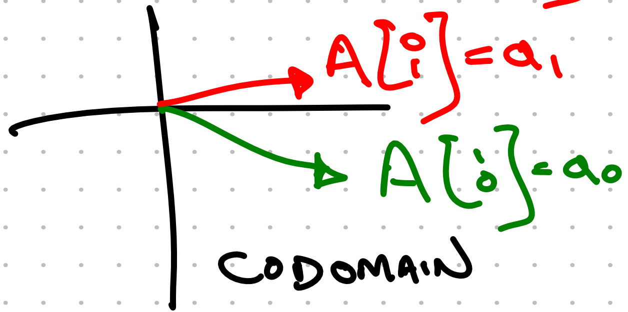
HOW TO BUILD ANY LINEAR FUNCTION ON VECTOR X

$$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$$

WHERE $a_0 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$a_1 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

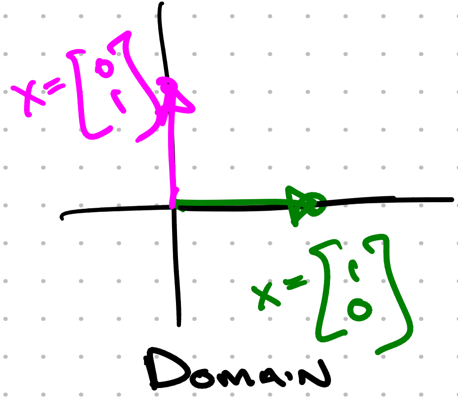


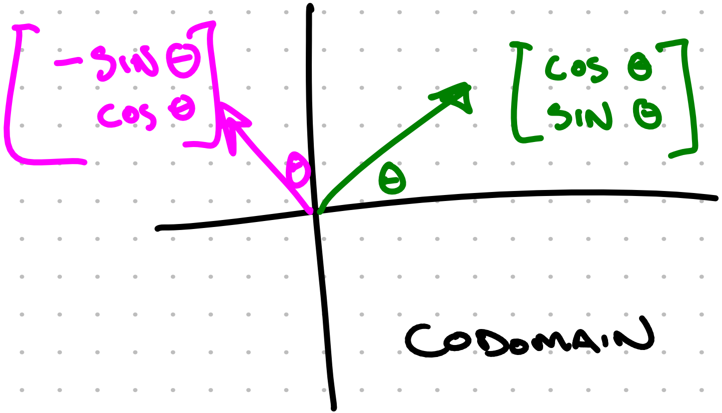


ROTATION ABOUT ORIGIN
IS A LINEAR TRANSFORM

Rotation Matrices

Rotate counter clockwise by θ





$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

MATRIX MULTIPLICATION IS ASSOCIATIVE

$$(AB)C = A(BC)$$

$$\begin{bmatrix} c_0 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 0 & c_2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = c_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} x_0 + c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} x_1$$

$$+ c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x_2$$

$$A = c\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D$$

$$BAx = Bcx$$

COMPOSING FUNCTIONS

$$ABx = A(Bx)$$

FIRST APPLY TRANSFORM B TO X

THEN APPLY TRANSFORM A TO X

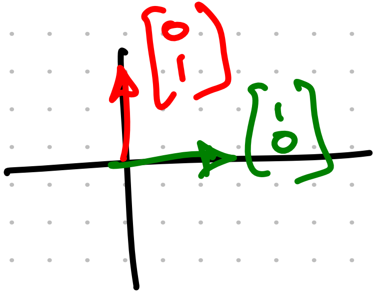
ICA 3

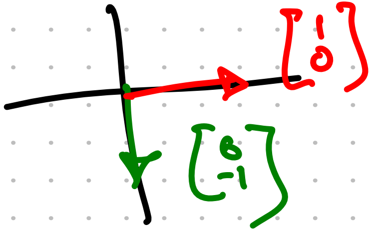
FIND A SINGLE MATRIX A WHICH WHEN MULTIPLIED BY $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ AS Ax

1 → ROTATES 2D VECTOR $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ CLOCKWISE ABOUT ORIGIN 90°

2 → SCALES x_1 BY -1

→ Rotates 2D vector $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ clockwise about origin 90°





$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

→ Scales x_1 by -1

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} x_1$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

BAx