CS2810 Day 3

Admin:
- ICA late policy
  - we don't take them late
  - we'll drop everyone's lowest ICA @ end of semester

Topics:

Matrices (and vectors)

Solution space of systems with many solutions

vocab: Singular Matrix / Homogenous Linear System

Length of vector

Dot Product & Angle between vectors

Matrices (and vectors)

A matrix is an array of scalars

$$\begin{array}{cc} & \text{3 columns} \\ \text{2 rows} & \begin{bmatrix}1 & 3 & 5 \\ 2 & 4 & 6\end{bmatrix}\end{array}$$

Matrix has shape 2×3

A vector is a matrix with 1 row or 1 column:

Column vector

$$\begin{bmatrix}1 \\ 2 \\ 3\end{bmatrix}$$

Shape: 3×1

Row vector

$$\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$$

Shape 1×4

MORE NOTATION AND CONVENTION

SCALARS - LOWERCASE, NOT BOLD $x = 2$

VECTORS - LOWERCASE, BOLD (COMPUTER)
ARROW HAT (HAND)

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

"TRULY 2D" MATRIX - UPPERCASE

NEITHER DIMENSION IS 1

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

MATRIX A HAS SHAPE 2×3 AND IS MADE OF REAL NUMBERS

↳ $A \in \mathbb{R}^{2 \times 3}$

A Few Matrix Operations

Matrix Addition

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Add corresponding entries of two matrices.

!!! matrices must have same shape !!!

Scalar Multiplication

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot 3 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \cdot 5 = \begin{bmatrix} 5 & 15 \\ 10 & 20 \end{bmatrix}$$

Multiply every entry of the matrix by some scalar.

Matrix Multiplication

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = ?$$

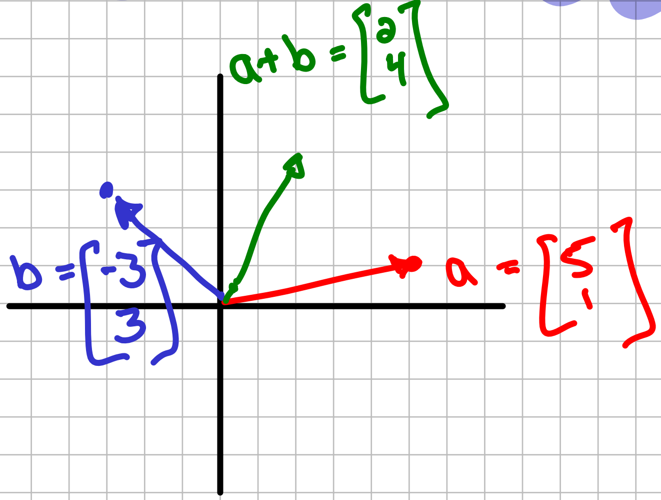
... Not yet

COMMON, NOT ENTIRELY UNIVERSAL, CONVENTION

⚠ ASSUME $\mathbb{R}^n$ IS A COL VECTOR

$x \in \mathbb{R}^4$ ASSUME $x = \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix}$ NOT $x = \begin{bmatrix} * & * & * & * \end{bmatrix}$

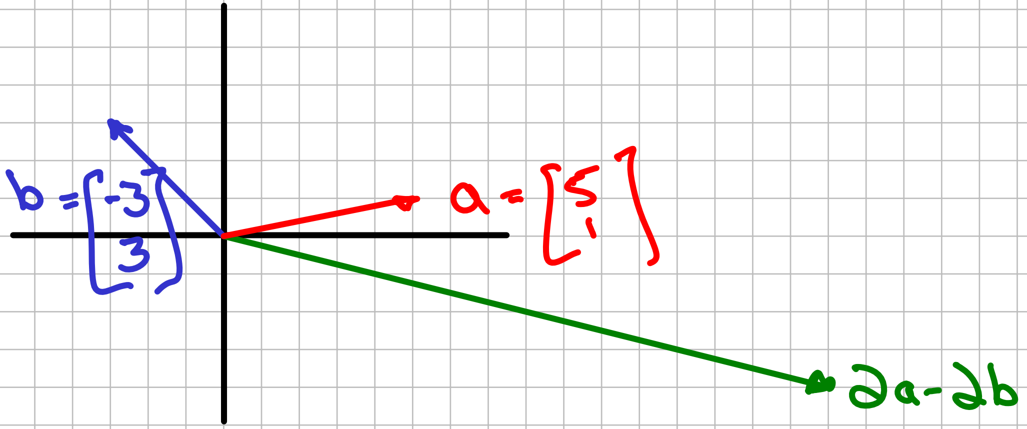
VECTOR ADDITION: "ONE SMALL STEP FOR YOUR BRAIN" ONE GIANT STEP FOR YOUR SKILLS



$$a+b = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$a+b$ IS THE RESULTING DISPLACEMENT AFTER TAKING ONE a STEP AND ONE b STEP

Vector Addition: "One small step for your brain" One giant step for your skills



$$2a-2b = 2\begin{bmatrix} 5 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 16 \\ -4 \end{bmatrix}$$

$2a$ $-2b$ is the resulting displacement after taking two a step and two backwards b steps

EXPRESSING SYSTEM SOLUTIONS VIA VECTOR EQN (MANY SOLUTION SYSTEMS)

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{l} x + z = 3 \\ y + 2z = 4 \\ z = z \end{array} \Rightarrow \begin{array}{l} x = 3 - 1z \\ y = 4 - 2z \\ z = 0 + 1z \end{array}$$

FOR EVERY 0 ON DIAGONAL, ADD CORRESPONDING REFLEXIVE EQUALITY (Z=Z OR SIMILAR)

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} z$$

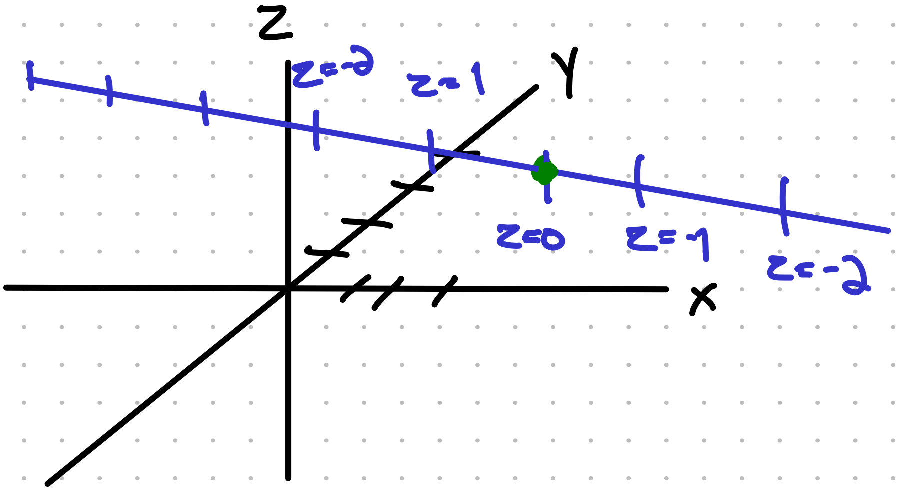
VISUALIZING SOLUTION SPACE (MANY SOLUTIONS)

WHAT ARE ALL THE $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ WHICH SATISFY:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} z$$

START HERE

ADD Z STEPS OF BLUE VECTOR



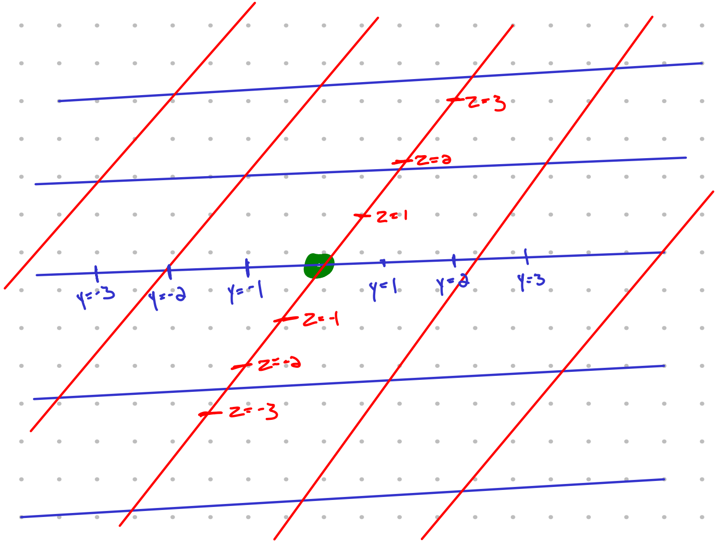
VISUALIZING SOLUTION SPACE (MANY SOLUTIONS)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ast \\ \ast \\ \ast \end{bmatrix} + \begin{bmatrix} \ast \\ \ast \\ \ast \end{bmatrix} y + \begin{bmatrix} \ast \\ \ast \\ \ast \end{bmatrix} z$$

START HERE

TAKE Y STEPS OF BLUE VECTOR

TAKE Z STEPS OF RED VECTOR



In general, solution space is N-dimensional if there are N "free parameters"
(y and z are "free" above, choose any value for them and we can find an x which satisfies equality)

Homogenous Systems

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

A system is homogenous if augment is all zeros

System

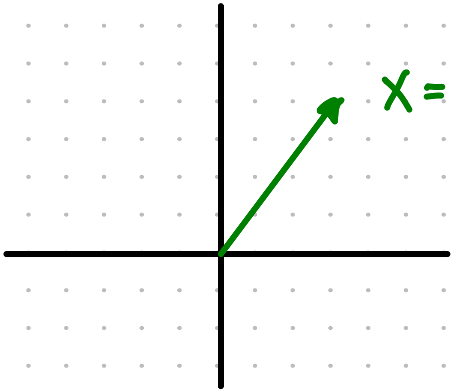
$$\left[\begin{array}{cc|c} 1 & 2 & * \\ 3 & 4 & * \end{array}\right]$$

Corresponding homogeneous system

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \end{array}\right]$$

HW

Vector Geometry: Length



$$x = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The length of x is

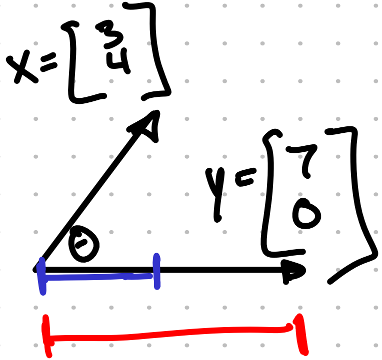
$$\|x\| = \sqrt{\sum_i x_i^2}$$

$$= \sqrt{3^2 + 4^2}$$

$$= \sqrt{9 + 16}$$

$$= 5$$

VECTOR GEOMETRY DOT PRODUCT INTUITION



DOT PRODUCT $x \cdot y$

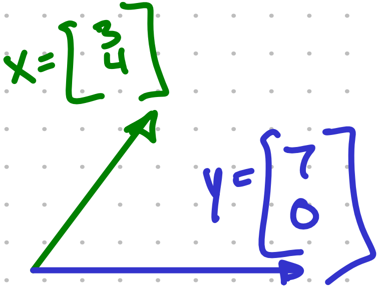
$=$ COMPONENT OF X IN DIRECTION OF Y $\times$ LENGTH OF Y

$$= \|x\| \cos\theta \times \|y\|$$

$$= 3 \times 7 = 21$$

$$x \cdot y = \|x\| \, \|y\| \cos\theta$$

Vector Geometry Dot Product Another way of computing



$$x \cdot y = \sum_i x_i \cdot y_i$$

$$= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

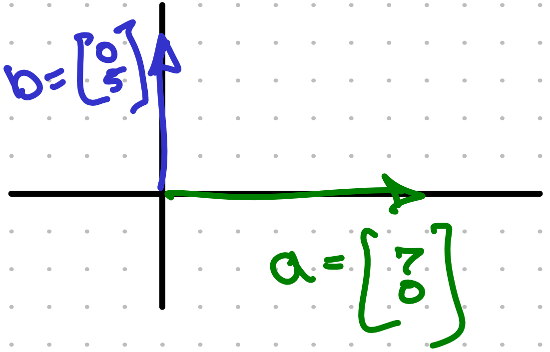
$$= 3 \cdot 7 + 4 \cdot 0 = 21$$

Multiply corresponding elements and sum results

VECTOR GEOMETRY DOT PRODUCT AND ANGLES

$$\sum x_i y_i = X \cdot Y = \|x\| \, \|y\| \cos\theta$$

WHAT'S THE ANGLE BETWEEN x, y?



$$\cos\theta = \frac{a \cdot b}{\|a\| \, \|b\|}$$

⚠ $X \cdot Y$ IS A SCALAR

(NOT A VECTOR)

DOT PRODUCT: WHY DO WE CARE?

→ EXTENDS OUR INTUITION OF ANGLES TO MORE (AND LESS) THAN 2D/3D SPACES

→ ALLOWS US TO IDENTIFY VECTORS AT RIGHT ANGLES TO EACH OTHER