CS2810 Feb 22

Day 11

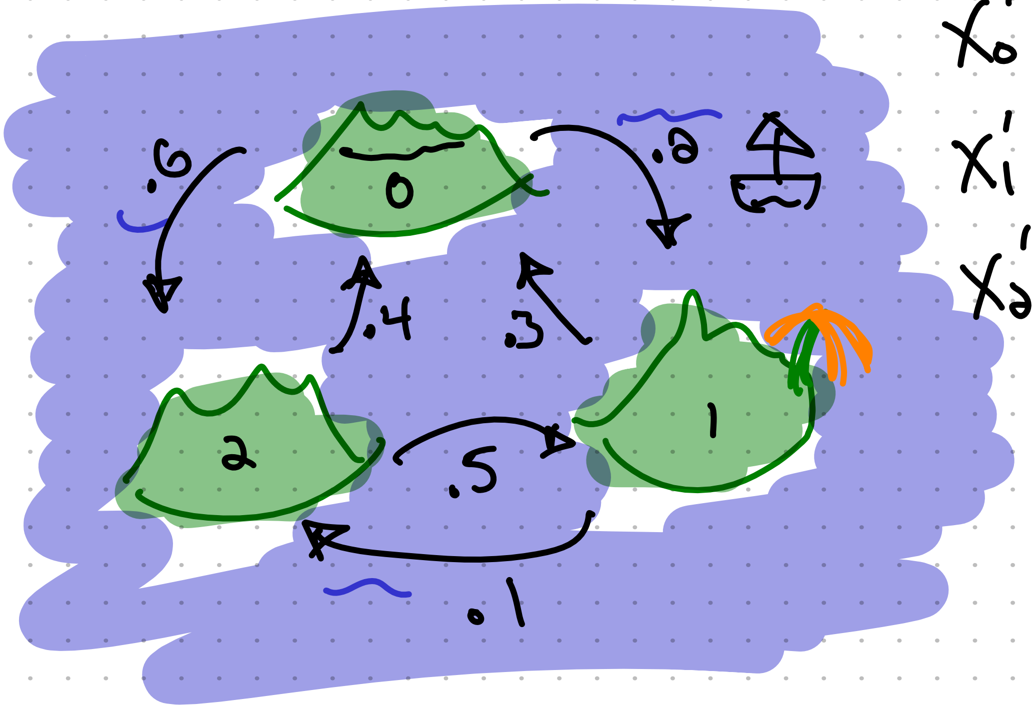
(Linear) Dynamical System FROM LAST CLASS
- ecology

Determinants

Eigenvalues & Eigenvectors
- Definition
- Finding eigenvalues
- Finding eigenvectors
- A matrix with distinct eigenvalues has linearly independent eigenvectors

Change of Basis and Eigenvectors

MATH "MAGIC" TRICK



\# PEOPLE ISLAND 0 AFTER 1 MOVE

$$x_0' = .2x_0 + .3x_1 + .4x_2$$

$$x_1' = .2x_0 + .6x_1 + .5x_2$$

$$x_2' = .6x_0 + .1x_1 + .1x_2$$

$$x_0' = .2x_0 + .3x_1 + .4x_2$$

$$x_1' = .2x_0 + .6x_1 + .5x_2$$

$$x_2' = .6x_0 + .1x_1 + .1x_2$$

x' POP AFTER MOVE

POPULATION ON EACH ISLAND

INIT X

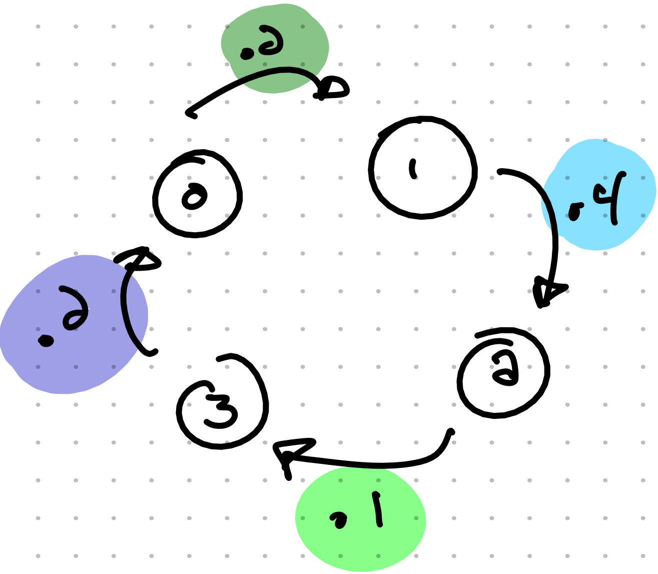
A

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} .2 \\ .2 \\ .6 \end{bmatrix} x_0 + \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix} x_1 + \begin{bmatrix} .4 \\ .5 \\ .1 \end{bmatrix} x_2 = \begin{bmatrix} .2 & .3 & .4 \\ .2 & .6 & .5 \\ .6 & .1 & .1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

$$x' = Ax$$

ICA 1

WRITE MATRIX A WHICH REPRESENTS ISLAND POPULATION AFTER 1 STEP



$$x_0' = x_0 - .2x_0 + .2x_3$$
$$x_1' = x_1 + .2x_0 - .4x_1$$
$$x_2' = x_2 + .4x_1 - .1x_2$$
$$x_3' = x_3 + .1x_2 - .2x_3$$

Dynamical System Question)

- Why/When does it have steady state distribution?

- How can I find steady state distribution?

(More later)

Determinant 2x2 CASE

A value associated with each square matrix:
- A matrix with det = 0 has linearly dependent columns
  (some non-zero linear combination of columns = zero vector) ✓

NOTATION

$$\mathrm{DET}(A) = |A| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = ad - bc$$

2x2 CASE

Determinant (3x3 case)

$$|A| = \left| \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right| = aei + bfg + cdh - afh - bdi - ceg$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

EIGENVALUES + EIGENVECTORS

An eigenvector is a non-zero vector, v, associated with a square matrix
- multiplying ~~Av scales v, but~~ doesn't change its direction

OUTPUT IS SCALED VERSION OF V

$$A\vec{v} = \lambda\vec{v}$$

A MULTIPLIES V

SCALE OF EIGENVECTOR

V IS ITS ASSOCIATED EIGENVALUE

$$A\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \lambda\begin{bmatrix}0\\0\end{bmatrix}$$

WHY $\vec{0}$ IS NOT EIG VEC

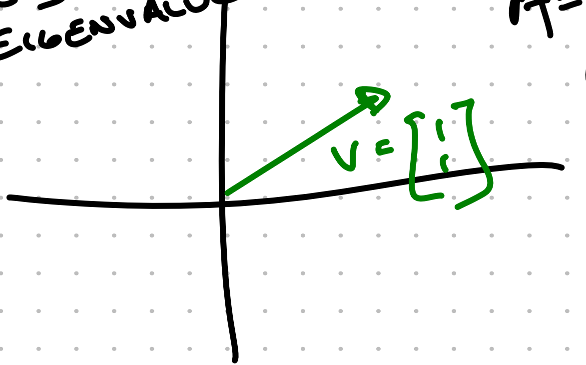
Geometry of Eigenvector | Eigenvalue

$v=\begin{bmatrix}1\\1\end{bmatrix}$ is eig vec w eigenvalue $\lambda=6$

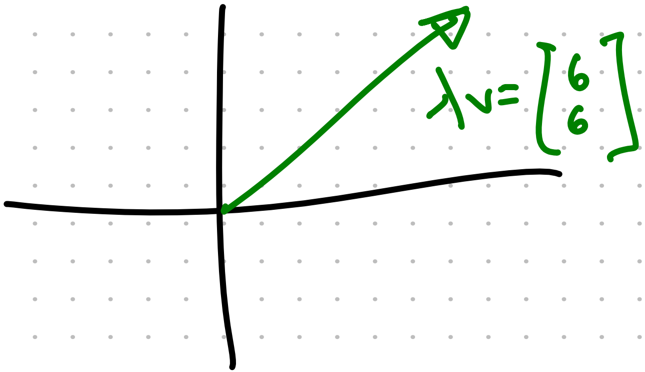
$A=\begin{bmatrix}1&5\\2&4\end{bmatrix}$

Notice:

$Av=\lambda v$ is in same direction as v



$Av=\lambda v$



Domain

Must be same dimension

Codomain

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = v \quad \text{AND} \quad \lambda = 6$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$$

$$Av = \lambda v$$

$$Av = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$\lambda v = 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

ICA 2:

Below are all the eigenvectors / eigenvalues of A. Match each eigenvector to its corresponding eigenvalue.

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$$

$\lambda = 6$ AND $\lambda = -1$

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ $v = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$

REMEMBER

EIG VEC / EIG VAL

PAIR / HAS

$Av = \lambda v$

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \qquad v = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \qquad \lambda = -1$$

$$Av = \lambda v$$

$$Av = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix} + \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$v\lambda = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \cdot -1 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$Av = \lambda v$$

$$A(vc) = c(Av) = c(\lambda v) = \lambda vc$$

SCALAR

Any scaled version of an eigenvector is also an eigenvector

(with same eigenvalue)

OBSERVE: EIGENVECTORS ASSOCIATED w/ SAME λ

NOT UNIQUE

$$\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad A\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

ANY SCALAR MULTIPLE OF EIG VEC IS ANOTHER EIG VEC w/ SAME λ

PUNCH LINE $\underline{Av = \lambda v}$

INSTED OF

$$Ax$$

TRY

$$A(c_0 v_0 + c_1 v_2) = c_0 A v_0 + c_1 A v_1$$
$$= c_0 \lambda_0 v_0 + c_1 \lambda_1 v_1$$

SINCE

$$x = c_0 v_0 + c_1 v_1$$

Finding Eigenvalues

Want: Method of finding λ, v pairs from A

Have: $Av = \lambda v$ $\rightarrow$

IDENTITY MATRIX

$$Av = \lambda I v$$

$$Av - \lambda I v = \vec{0}$$

$$(A - \lambda I)v = \vec{0}$$

Some non-zero linear combo of columns of this matrix is the zero vector

Want: λ with $\det(A - \lambda I) = 0$

FIND EIGENVALUES OF $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 5 \\ 2 & 4-\lambda \end{bmatrix}$$

$$0 = \det(A - \lambda I) = (1-\lambda)(4-\lambda) - 2 \cdot 5$$

$$= 4 - 4\lambda - \lambda + \lambda^2 - 10$$

$$= \lambda^2 - 5\lambda - 6$$

$$= (\lambda - 6)(\lambda + 1)$$

$\lambda = 6$ OR $\lambda = -1$

EVERY $N \times N$ MATRIX HAS N EIGENVALUES

SINCE

EVERY POLYNOMIAL OF ORDER N HAS N ROOTS

FIND CORRESPONDING EIGENVECTOR $A = \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$

$\lambda = -1$ OR 6

$(A - \lambda I)v = 0$

$(A - (-1)I)v = 0$ $\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} 10 + \begin{bmatrix} 5 \\ 5 \end{bmatrix} -4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left(\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) v = 0$

$v = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$ OR $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

Finding an eigenvalue of square matrix:

- Find roots of "characteristic polynomial"

SOLVE $\det(A - \lambda I) = 0$

Finding an eigenvector associated with eigenvalue:

SOLVE $(A - \lambda I)v = \vec{0}$

ICA FIND ALL EIGENVECTOR / EIGENVALUE PAIRS

OF $A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$

$$\text{DET}\left(\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0 = \text{DET}\left(\begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix}\right)$$

$$0 = (7-\lambda)(-1-\lambda) - 9$$

$$= -7 + \lambda - 7\lambda + \lambda^2 - 9 = \lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$$

$$(A - \lambda I)v = \vec{0}$$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix}$$

$\lambda = 8$ or -2

$$\left(\begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \right) v = \vec{0}$$

$$v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 9 \\ 3 \end{bmatrix} + -3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} \rightarrow (A - \lambda I)v = 0 \rightarrow \begin{bmatrix} 7-8 & 3 \\ 3 & -1-8 \end{bmatrix} v = 0$$

$$\rightarrow \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} v = 0 \rightarrow v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ WAS } \lambda = 8$$

Numpy

HOW DOES THE MAGIC TRICK WORK?

$$\begin{bmatrix} x_0' \\ x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .2 & .6 & .5 \\ .6 & .1 & .1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

POP AFTER MOVE

POP BEFORE MOVE

INSIGHT: WRITE X AS A LINEAR COMBO OF EIG VECS OF A

$$X = C_0 V_0 + C_1 V_1 + C_2 V_2$$

$$\text{EIG VEC} @ C = X$$

$$C = (\text{EIG VEC})^{-1} X$$

$$A^{100}x = A^{100}\left(c_0 v_0 + c_1 v_1 + c_2 v_2\right)$$

$$= c_0 \underline{A^{100} v_0} + c_1 \underline{A^{100} v_1} + c_2 \underline{A^{100} v_2}$$

$$= c_0 \lambda_0^{100} v_0 + c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2.$$

$$Ax = A\left(c_0 v_0 + c_1 v_1 + c_2 v_2\right)$$

IS THIS GOOD FOR ANYTHING BESIDES "MAGIC"?

"MATRIX DIAGONALIZATION" IS COMPUTATIONALLY EFFICIENT

$$A^{100}x = c_0 \lambda_0^{100} v_0 + c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2$$

100 MATRIX MULTIPLICATIONS

ONE EXPONENT COMPUTE