CS 2810 Day 2 Jan 21

Admin:

- ICA process reminder
    - ICAs not included in notes on website
    - upload all ICAs in a lesson to one assignment on gradescope

Linear Systems
Representing Linear System as augmented matrix
RREF
identifying solutions of linear systems (none, unique, many)

LINEAR SYSTEM IS A SET OF LINEAR EQUALITIES

EX:

$$x + y = 0$$
$$2x - y + 3z = 3$$
$$x - 2y - z = 3$$

SOLUTIONS OF SYSTEM MUST RESOLVE ALL EQUALITIES

EQUIVALENT LINEAR SYSTEM 1

By scaling and adding any row (equation) to any other row, we don't change system's solutions.

SYSTEM A HAS THE SAME SET OF SOLUTIONS AS SYSTEM B

System A:

$r_0: x + y = 0$

$r_1: -2x = 4$

System B:

$r_0: x + y = 0$

$r_1: 2y = 4$

NEW r_1 IS OLD r_1 PLUS $2r_0$ $(r_1' = r_1 + 2r_0)$

SYSTEM A HAS THE SAME SET OF SOLUTIONS AS SYSTEM B

$r_0: x + y = 0$

$r_1: -2x = 4$

$(r_1' = r_1 + 2r_0)$

$r_0: x + y = 0$

$r_1: 2y = 4$

$$r_1 \xrightarrow{r_1} -2x = 4$$

$$r_0 \xrightarrow{2r_0} 2x + 2y = 0$$

$$\overline{\quad 2y = 4 \quad} \xrightarrow{r_1'} r_1$$

EQUIVALENT LINEAR SYSTEM 2

By scaling a row (equation), we don't change solution set of the system.

SYSTEM B HAS THE SAME SET OF SOLUTIONS AS SYSTEM C

SYSTEM B

$r_0: x+y = 0$

$r_1: 2y = 4$

SYSTEM C

$r_0: x+y = 0$

$r_1: y = 2$

NEW r_1 IS $1/2$ OLD r_1 $\left(r_1' = 1/2\, r_1\right)$

SOLVING A LINEAR SYSTEM

$$x + y = 0$$
$$2x - y + 3z = 3$$
$$x - 2y - z = 3$$

$\rightsquigarrow$

ENOUGH ROW OPERATIONS

$$x = 1$$
$$y = -1$$
$$z = 0$$

(More Detail)

$$\begin{aligned} x+y &= 0 \\ 2x-y+3z &= 3 \\ x-2y-z &= 3 \end{aligned} \quad \xrightarrow[r_2' = r_2 - r_0]{r_1' = r_1 - 2r_0} \quad \begin{aligned} x+y &= 0 \\ -3y+3z &= 3 \\ -3y-z &= 3 \end{aligned} \quad \xrightarrow{r_1' = -1/3\, r_1} \quad \begin{aligned} x+y &= 0 \\ y-z &= -1 \\ -3y-z &= 3 \end{aligned}$$

$$\xrightarrow[r_2' = r_2 + 3r_1]{r_0' = r_0 - r_1} \quad \begin{aligned} x+z &= 1 \\ y-z &= -1 \\ -4z &= 0 \end{aligned} \quad \xrightarrow{r_2' = -1/4\, r_2} \quad \begin{aligned} x+z &= 1 \\ y-z &= -1 \\ z &= 0 \end{aligned} \quad \xrightarrow[r_1' = r_1 + r_2]{r_0' = r_0 - r_2} \quad \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 0 \end{aligned}$$

HOW DO YOU CHOOSE ROW OPERATIONS TO SOLVE SYSTEM?

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

GOAL

$$\begin{aligned} x &= 1 \\ y &= -1 \\ z &= 0 \end{aligned} \iff \begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= -1 \\ 0x + 0y + 1z &= 0 \end{aligned}$$

CHOOSE ROW OPERATIONS WHICH INDUCE A SINGLE ONE AND ZEROS OTHERWISE, PER ROW

(we'll provide a more formal method later ...)

ICA A

Solve the linear system by row reduction, document your row operations as shown previously (e.g. r_1' = r_1 + 4 r_0)

$$\begin{aligned} 4x - y &= 6 \\ 2x + y &= 0 \end{aligned} \xrightarrow{r_0' = r_0 + r_1} \begin{aligned} 6x &= 6 \\ 2x + y &= 0 \end{aligned} \xrightarrow{r_0' = 1/6\, r_0} \begin{aligned} x &= 1 \\ 2x + y &= 0 \end{aligned}$$

$$\begin{array}{ll} r_0 & 4x - y = 6 \\ r_1 & 2x + y = 0 \\ \hline r_0' & 6x = 6 \end{array}$$

$$\begin{aligned} x &= 1 \\ 2x + y &= 0 \end{aligned} \quad \xrightarrow{r_1' = r_1 - 2r_0} \quad \begin{aligned} x &= 1 \\ y &= -2 \end{aligned}$$

$$\begin{array}{rrcl} -2r_0: & -2x & & = -2 \\ r_1: & 2x & + y & = 0 \\ \hline & & y & = -2 \end{array}$$

Matrices: Just a Notation (for now)

To help us avoid rewriting x y z

$$
\begin{array}{rcc}
1x + 1y + 1z = 9 & \leftarrow r_0 \rightarrow & \\
0x + 1y - 1z = 2 & \leftarrow r_1 \rightarrow & \\
1x + 1y + 3z = 9 & \leftarrow r_2 \rightarrow &
\end{array}
\qquad
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\left[\begin{array}{ccc|c}
1 & 1 & 1 & 9 \\
0 & 1 & -1 & 2 \\
1 & 1 & 3 & 9
\end{array}\right]
\end{array}
$$

"Augment column"

Whole thing is Augmented Matrix

This 3x3 array is Matrix

⚠ Distinguish Matrix vs Augmented Matrix

ICA B

Solve the linear system by row reduction

- document your row operations as previously shown
- represent it as a matrix at each step

$$\left[\begin{array}{cc|c} 1 & 0 & ? \\ 0 & 1 & ? \end{array}\right] \quad \begin{array}{l} x = ? \\ y = ? \end{array}$$

$$2x + y = 1$$
$$-x + y = 0$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{r_0' = \frac{1}{2} r_0} \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow[r_1' = r_1 + r_0]{} \left[\begin{array}{cc|c} 1 & 1/2 & 1/2 \\ 0 & 3/2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \xi \\ 0 & 1 & \varepsilon \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow[r_1' = \frac{2}{3} r_1]{} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \end{array}\right] \xrightarrow{r_0' = r_0 - \frac{1}{2} r_1} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{array}\right]$$

$$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{6} \end{matrix}$$

$$1x + 0y = \tfrac{1}{3} = x$$
$$0x + 1y = \tfrac{1}{3} = y$$

ICA 3

Solve the linear system by row reduction

- document your row operations as previously shown
- represent it as a matrix at each step

$$x + y = 1$$
$$-2x + y = 1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 1 & 1 \end{array}\right] \xrightarrow{r_1' = r_1 + 2r_0} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{r_1' = 1/3\, r_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{r_0' = r_0 - r_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} 1x + 0y = 0 \\ 0x + 1y = 1 \end{array}$$

$$\Rightarrow \begin{array}{l} x = 0 \\ y = 1 \end{array}$$

REDUCED ROW ECHELON FORM (RREF)

(THE "SIMPLEST" FORM OF AN AUGMENTED MATRIX)

WE USE ξ TO INDICATE ANY NUMBER

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & \xi \end{array}\right]$$

REDUCED ROW ECHELON FORM (RREF)

(THE "SIMPLEST" FORM OF AN AUGMENTED MATRIX)

WE USE ξ TO INDICATE ANY NUMBER

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \xi \\ 0 & 1 & 0 & \xi \\ 0 & 0 & 1 & \xi \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \xi \\ 0 & 1 & \xi \\ 0 & 0 & \xi \\ 0 & 0 & \xi \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \xi & \xi & \xi \\ 0 & 0 & 0 & \xi \end{array}\right]$$

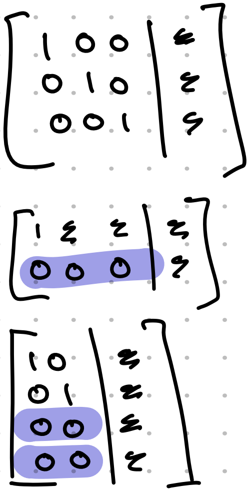
CAN BE ANY VALUE

WE REFER TO THESE ROWS AS 'ZERO ROWS'

ALL THESE AUGMENTED MATRIX ARE IN RREF

REDUCED ROW ECHELON FORM (RREF)

(THE "SIMPLEST" FORM OF AN AUGMENTED MATRIX)

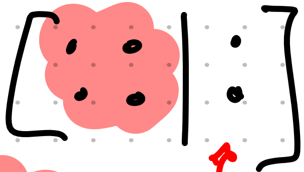


RREF REQUIREMENTS

→ ALL 'ZERO ROWS' AT BOTTOM

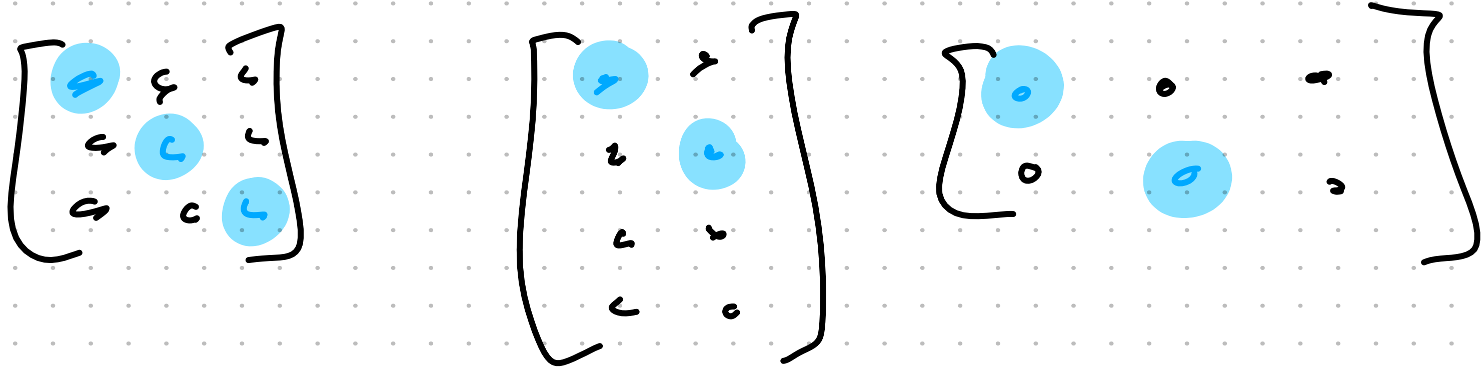
→ 1 ON DIAGONAL OF REMAINING MATRIX

→ 0 ABOVE | BELOW ALL THESE 1'S ON DIAGONAL



NOT INCLUDED

MATRIX ANATOMY: DIAGONAL

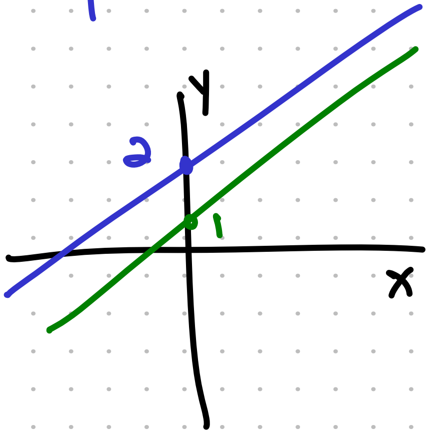


Types of Linear Systems Solutions

No solution

$y = x + 1$

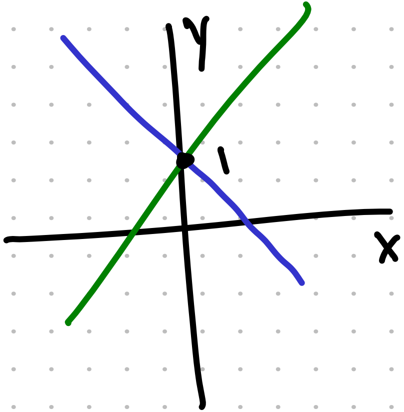
$y = x + 2$



Unique solution

$y = x + 1$

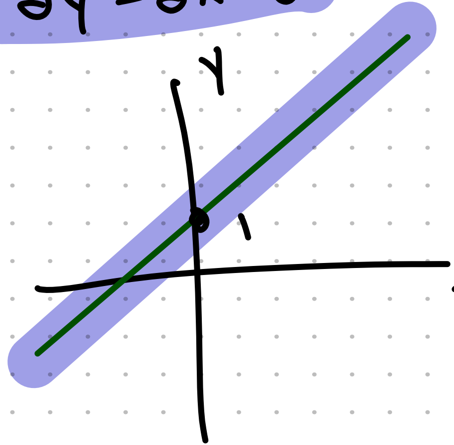
$y = -x + 1$



Many solutions

$y = x + 1$

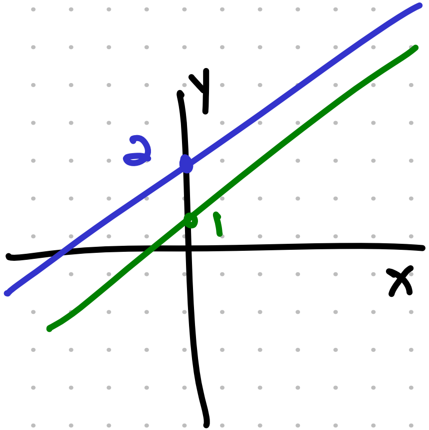
$2y = 2x + 2$



TYPES OF LINEAR SYSTEMS SOLUTIONS (RREF)

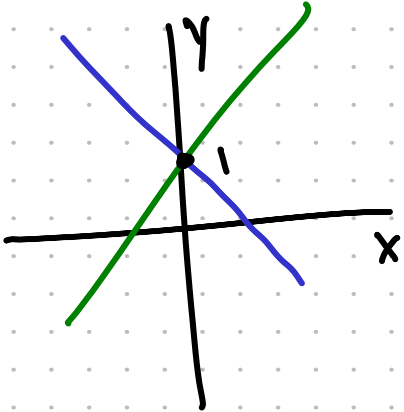
NO SOLUTION

RREF $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$



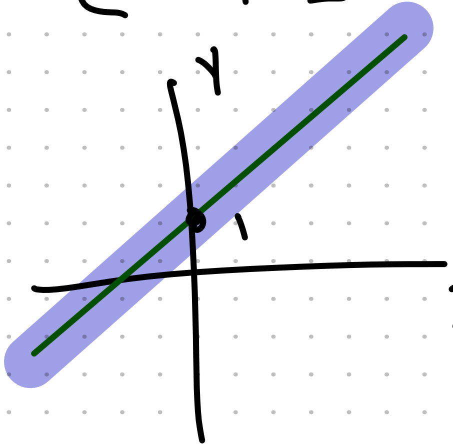
UNIQUE SOLUTION

RREF $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$



MANY SOLUTIONS

RREF $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right]$



Types of Linear Systems Solutions (RREF Motivate)

No Solution

RREF $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

$x - y = 0$

$0x + 0y = 1 \rightarrow 0 = 1$

Not possible for any x, y $\rightarrow\leftarrow$

Unique Solution

RREF $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$

$x = 0$

$y = 1$

Many Solutions

RREF $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

$x - y = -1$

$0 = 0$

Always true

Types of Linear Systems Solutions (RREF Anatomy)

No Solution

RREF $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

→ Has zero row w/ non-zero augment

Unique Solution

RREF $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right]$

→ RREF w/ as many 1's as columns in matrix

Many Solutions

RREF $\left[\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

→ Has a row in matrix with more than 1 non-zero entry

→ All zero rows have zero augment

ICA C

Tell whether each matrix below is in RREF
For any matrices not in RREF, cite a particular entry which justfies its exclusion
For any matrices in RREF, tell whether they have

- no solutions
- unique solution
- many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 4 \end{array}\right]$$

is RREF
no solutions
last row implies 0 =4

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 8 \end{array}\right]$$

2 in matrix makes it
not RREF.
0s above/below 1s
in the matrix for RREF

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

one 1 per row implies
x = squiggle_1
y = squiggle_2 ...

a unique solution

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

many solutions
0 =0 is always true
(last row doesn't
constrain solutions)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & [illegible] & 2 \\ 0 & 1 & 4 & 5 \end{array}\right] \end{array}$$

$0 \implies 0 = 0$ MANY

NON-ZERO $\implies 0 = 1$ NO SOLUTIONS

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & u \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & \end{array}\right]$$

Every linear system (except those with no solutions) has a unique RREF

The existance of a Zero row means that system has no solutions or many solutions.

Student misconception:
A 'zero row' requires a zero in the augment column: False

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 3 \end{array}\right]$$

NEXT LECTURE

Solving all linear systems: bring them to RREF

1. Express linear system as an augmented matrix:

$$\left[\begin{array}{cc|c} 4 & 8 & 12 \\ -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & 6 & 0 & -9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2. Scale row i so that its leading coefficient is 1

FIRST NON-ZERO ENTRY IN ROW (NOT INCLUDING AUGMENT)

$$\left[\begin{array}{cc|c} 4 & 8 & 12 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{r_0' = 1/4\, r_0} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 1 & 1 \end{array}\right]$$

3. Add & scale row i to all other rows so matrix has only 0 above/below leading coefficient

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{r_1' = r_1 + r_0} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 3 & 4 \end{array}\right]$$

4. Repeat for row i + 1 if it has a leading coefficient

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 5 \end{array}\right]$$