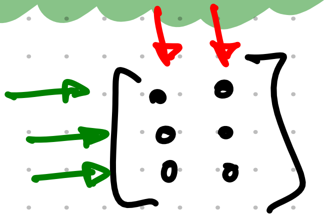
CS2810 Day 6 Feb 4

- Matrix-Matrix Multiplication
- Linear Combinations
- Matrix Multiplication as a Function (Transformation)
- Building Matrix Function from Linearity
  - Scaling
  - Rotating
- Composing Matrix Functions $BAx = y$

Admin

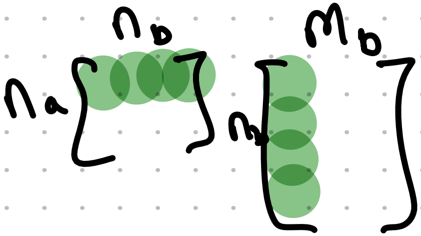
→ Notes format

Matrix-Matrix Multiplication: Shape Rule



has shape $\overset{\text{N rows}}{3} \times \overset{\text{N col}}{2}$

Let A has shape $n_a \times m_a$
B has shape $n_b \times m_b$



AB is only defined if $m_a = n_b$

if defined AB has shape $n_a \times m_b$

SHAPE RULE EXAMPLE

A HAS SHAPE 10×17

B HAS SHAPE 17×14

GIVE OUTPUT SHAPE OF EACH OPERATION

AB

$$(10 \times 17)(17 \times 14)$$

AB HAS SHAPE (10×14)

BA

$$(17 \times 14)(10 \times 17)$$

NOT DEFINED

Transpose of a matrix swaps rows / cols

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

2×3 $\qquad$ 3×2

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad x^T = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

3×1 $\qquad$ 1×3

IS $AB = BA$ FOR ALL MATRICES

FALSE

MATRIX MULTIPLICATION IS NOT COMMUTATIVE

ICA 1

GIVE SHAPE OF EACH MATRIX PRODUCT IF IT EXISTS

$\text{SHAPE}(A) = 3 \times 3$ $\qquad$ $\text{SHAPE}(B) = 3 \times 1$

$\text{SHAPE}(C) = 3 \times 4$ $\qquad$ $\text{SHAPE}(D) = 1 \times 4$

AB

$(3 \times \cancel{3})(\cancel{3} \times 1)$

3×1

CA

$(3 \times 4)(3 \times 3)$

NOT DEFINED

AC

$3 \times \cancel{3}\ \cancel{3} \times 4$

3×4

$(D C^T) B$

$(1 \times \cancel{4})(\cancel{4} \times 3)(3 \times 1)$

$(1 \times 3)(3 \times 1)$

1×1

Matrix-Matrix Multiplication: Computing (Dot Product)

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \\ 1 & \\ & \end{bmatrix}$$

$1 \cdot 3 + 0 \cdot -2 = 3$

$1 \cdot 3 + 1 \cdot -2 = 1$

$2 \cdot 3 + 7 \cdot -2 =$

Each element in Product Matrix is Dot Product of corresponding Row (Left Matrix) and Col (Right Matrix)

LINEAR COMBINATION (WEIGHTED SUM)

COULD BE ANY OBJECTS, MOST OFTEN MATRICES FOR US

A LINEAR COMBINATION OF $\overbrace{X_0, X_1, X_2}, \ldots$

IS $\alpha_0 X_0 + \alpha_1 X_1 + \alpha_2 X_2 + \ldots$

WHERE EACH α_i ARE SCALARS

MATRIX - VECTOR MULTIPLICATION

(IN THAT ORDER)

$$Ax = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 9 \\ 1 \cdot 10 + 2 \cdot 11 + 3 \cdot 12 \end{bmatrix}$$

A $\quad x$

MATRIX VECTOR MULTIPLICATION GIVES A LINEAR COMBINATION OF COLUMNS OF MATRIX $= 1 \cdot \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 4 & 5 & 6 & 11 \\ 7 & 8 & 9 & 12 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}}_{X} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

$$x_0 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_1 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

ICA 1.5

BUILD THE MATRIX A WHICH, WHEN MULTIPLIED AS Ax, ALWAYS YIELDS THE SAME VECTOR x

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = A \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

WHAT IS A?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

HINT: NOTATION FOR COLUMNS WE DON'T KNOW YET...

$$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = A\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = x_0\begin{bmatrix} | \\ a_0 \\ | \end{bmatrix} + x_1\begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}$$

$$(2\times 1) = (2\times \cancel{2})(\cancel{2}\times 1)$$

$$= x_0\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

MULTIPLICATIVE IDENTITY

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

VECTOR-MATRIX MULTIPLICATION

IN THAT ORDER

$$xA = \underset{(1,3)}{\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}} \underset{(3,3)}{\begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}} = \overset{(1,3)}{\begin{bmatrix} 1\cdot 4 + 2\cdot 7 + 3\cdot 10, & 1\cdot 5 + 2\cdot 8 + 3\cdot 11, & 1\cdot 6 + 2\cdot 9 + 3\cdot 12 \end{bmatrix}}$$

xA

VECTOR-MATRIX MULTIPLICATION IS A LINEAR COMBO OF ROWS OF A

$$= 1\cdot\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} + 2\cdot\begin{bmatrix} 7 & 8 & 9 \end{bmatrix} + 3\cdot\begin{bmatrix} 10 & 11 & 12 \end{bmatrix}$$

TRANSPOSE IDENTITIES AND HOW THEY RELATE MATRIX-VECTOR TO VECTOR-MATRIX MULTIPLICATIONS

$$(A^T)^T = A$$

$$(xA)^T = A^T x^T$$

$$(AB)^T = B^T A^T$$

$$(Ax)^T = x^T A^T$$

MATRIX-VECTOR AND VECTOR MATRIX MULTIPLICATIONS ARE A TRANSPOSE AWAY FROM EACH OTHER

CONVENTION:

PREFER MATRIX-VECTOR MULTIPLICATION WHERE POSSIBLE (COL VECTORS)

(AVOID VECTOR-MATRIX W/ ROW VECTORS)

ICA 2

Simplify Expressions Below

$$\begin{bmatrix} 1 & 0 & -4 \\ 1 & 0 & -4 \\ 1 & 0 & -4 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} -4 \\ -4 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} -11 \\ -11 \\ -11 \end{bmatrix} = -11\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(3\times1)\ (1\times3)$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

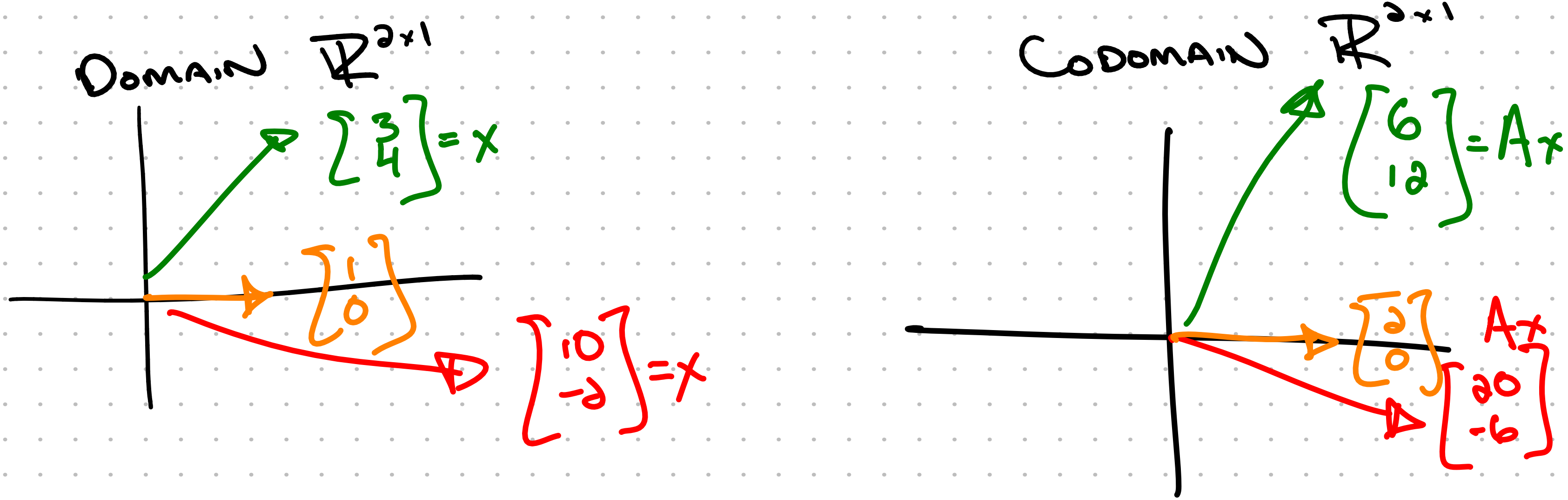
$$1/3\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 1/3\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1/3\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Matrix-Vector Multiplication as a Function

Let $A \in \mathbb{R}^{2\times 2}$ consider $f: \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$

$$f(x) = Ax = b$$

Given a 2d column vector x, find the matrix A which, when multiplied as Ax, yields an output which:
- doubles the first dimension of x
- triples the second dimension of x



BUILDING TRANSFORM A

LET a_0, a_1 BE COLUMNS OF $A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$ $\quad A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

LET $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ WANT A WITH $\begin{bmatrix} 2 \\ 0 \end{bmatrix} = A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} | \\ a_0 \\ | \end{bmatrix}\cdot 1 + \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}\cdot 0$

LET $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ WANT A WITH $\begin{bmatrix} 0 \\ 3 \end{bmatrix} = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} | \\ a_0 \\ | \end{bmatrix}\cdot 0 + \begin{bmatrix} | \\ a_1 \\ | \end{bmatrix}\cdot 1$

BUT YOU JUST BUILT $\underline{A}$ FROM ONLY TWO INPUT VECTORS, HOW DO YOU KNOW IT WORKS FOR OTHERS?

MATRIX MULTIPLICATION (BY A) IS LINEAR

$$A(\alpha x + \beta y) = \alpha A x + \beta A y$$

BUT YOU JUST BUILT A FROM ONLY TWO INPUT VECTORS. HOW DO YOU KNOW IT WORKS FOR OTHERS?

$x = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$ EXPECT $Ax = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$

$$A\begin{bmatrix} 7 \\ -2 \end{bmatrix} = A\left(7 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 7A\begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2A\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 7\begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

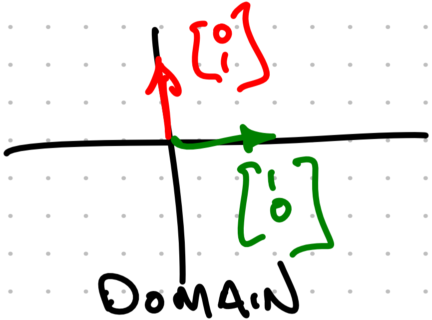
$$= \begin{bmatrix} 14 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ -6 \end{bmatrix}$$

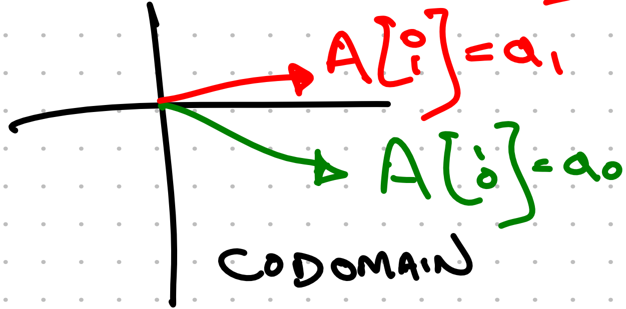
How to Build ANY Linear Function on Vector x

$$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$$

WHERE $a_0 = A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$a_1 = A \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

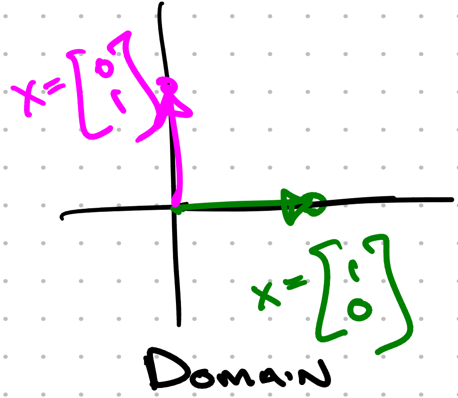


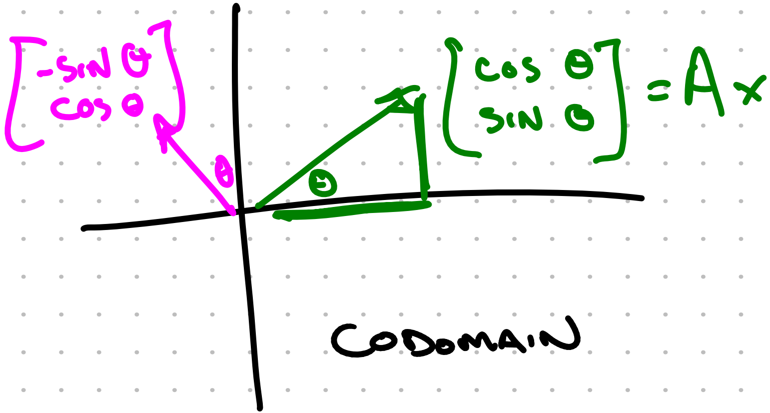


ROTATION ABOUT ORIGIN
IS A LINEAR TRANSFORM

Rotation Matrices

Rotate counter clockwise by θ





$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

MATRIX MULTIPLICATION IS ASSOCIATIVE

$$(AB)C = A(BC)$$

$$ABC$$

Composing Functions

$$ABx = A(Bx)$$

First apply transform B to x

Then apply transform A to x

ICA 3

Find a single matrix A which when multiplied by $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ as Ax

$A \rightarrow$ Rotates vector $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$ clockwise about origin $90°$

$B \rightarrow$ Scales x_1 by -1

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$A = \begin{bmatrix} 0 & \\ -1 & \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\underline{BA}\, x$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Answer