As you get settled...

an acceptable answer:
"I'm going to sign-up after class"
⇓
but actually do it !!

- Get out a place to do today's ICA (8)
  - remember to write your name, my name, the ICA #, and the date!

ICA question 0: sign into khouryofficehours.com and navigate to our course's page. Did you do this/can you do this successfully?

Write down the available office hours over the weekend.

↳ we're only up to 55% !!

Sign into piazza. What example did Prof. Felix post to help you all with HW 2?

- You can sign up by visiting www.piazza.com/northeastern/spring2022/cs2810sp2022 and using the access code `cs2810_pass`

Projections & Lines of best fit: Or, how to stop drawing in 3 dimensions and draw in 2 or 1 instead

Projections

- A projection is a matrix transformation that we can apply to a point as many times as we'd like and always get the same result out

vector

- $Ax = b$ output

- $AAx = b$ same b

- $AAAx = b$

$A^2 = A$

- For example: $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ $\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\left(\begin{bmatrix}2\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$$

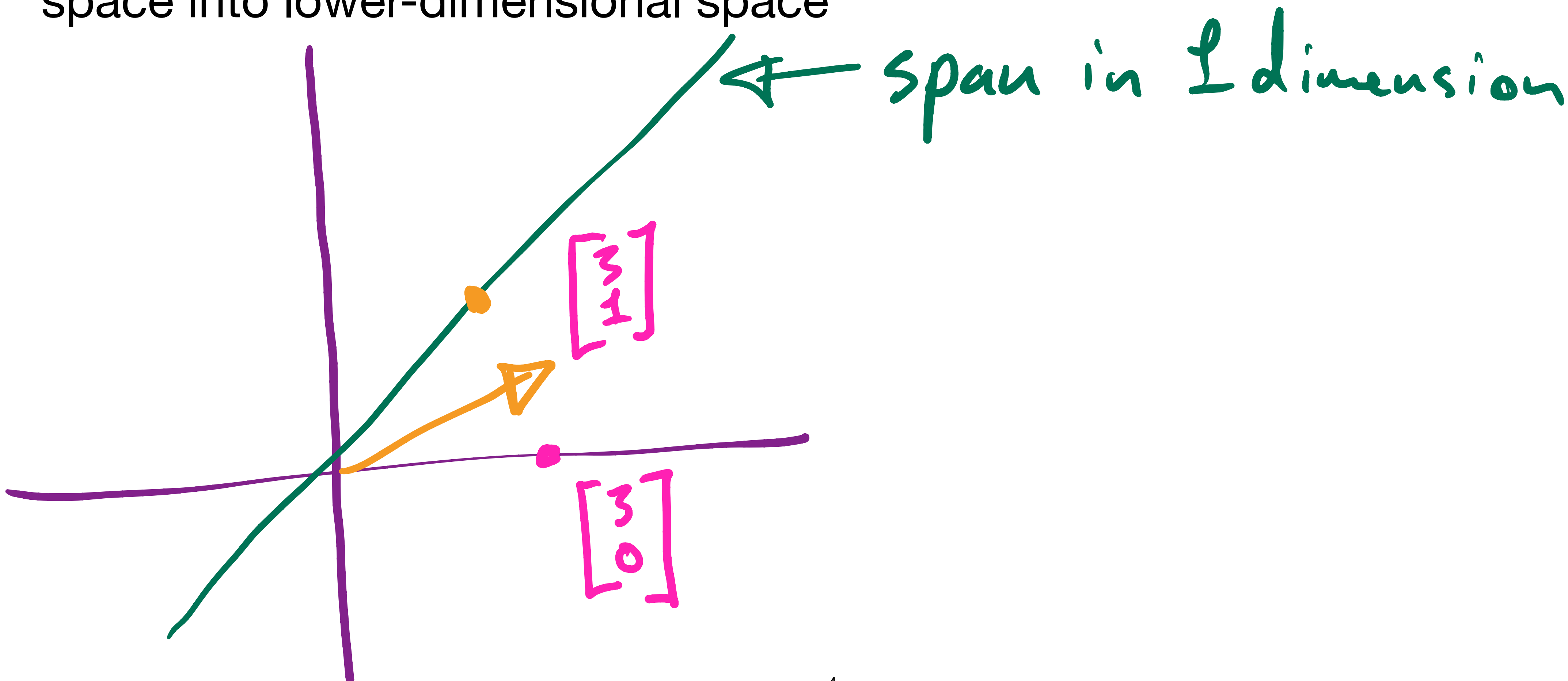
not a projection:

$$\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}4 & 0\\0 & 0\end{bmatrix}$$

Projections

- Why?

- We'll use projection matrices to project vectors in higher-dimensional space into lower-dimensional space



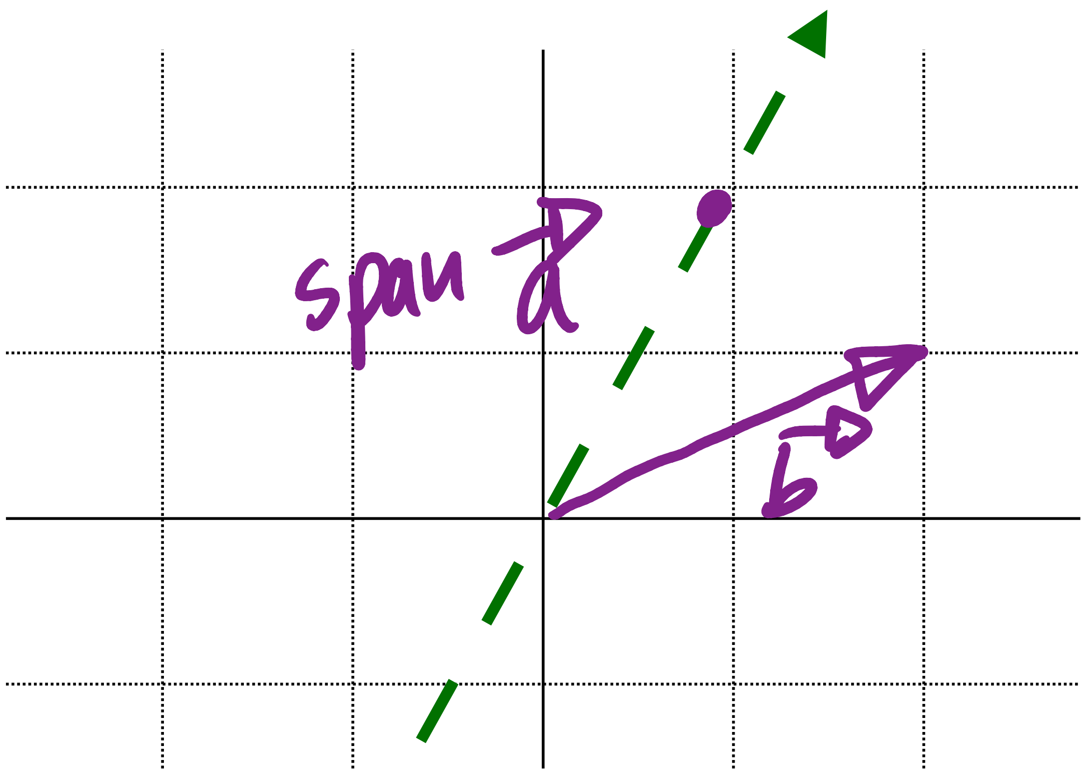
Projections

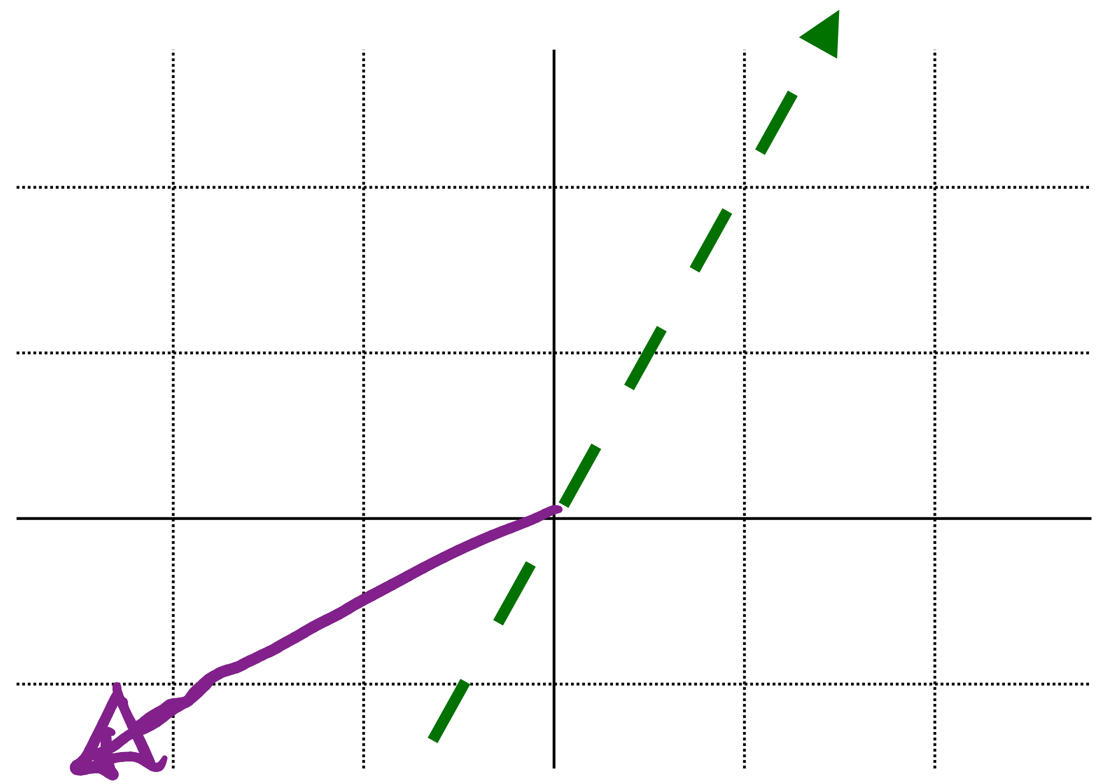
- Say you have a vector space defined by the following vectors:
- $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$
- What is the span of $\{ \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \}$? $\Rightarrow$ a line $2x = y$

Projections

- What is the span of $\left\{\begin{bmatrix}-1\\-2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right\}$?

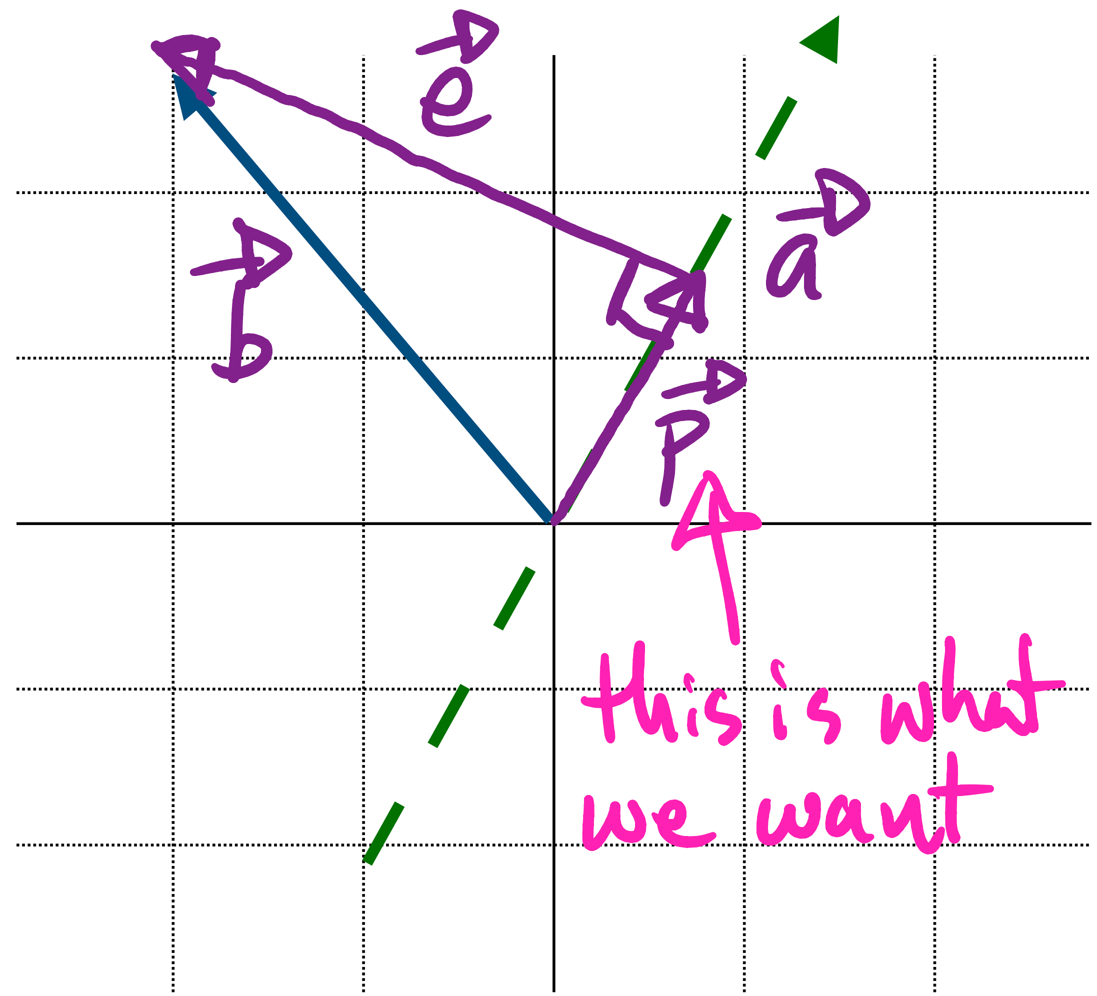
- Goal: given any new vector $\vec{b}$, find the point in the span of $\vec{a}$ that is closest to $\vec{b}$





Projections onto a line

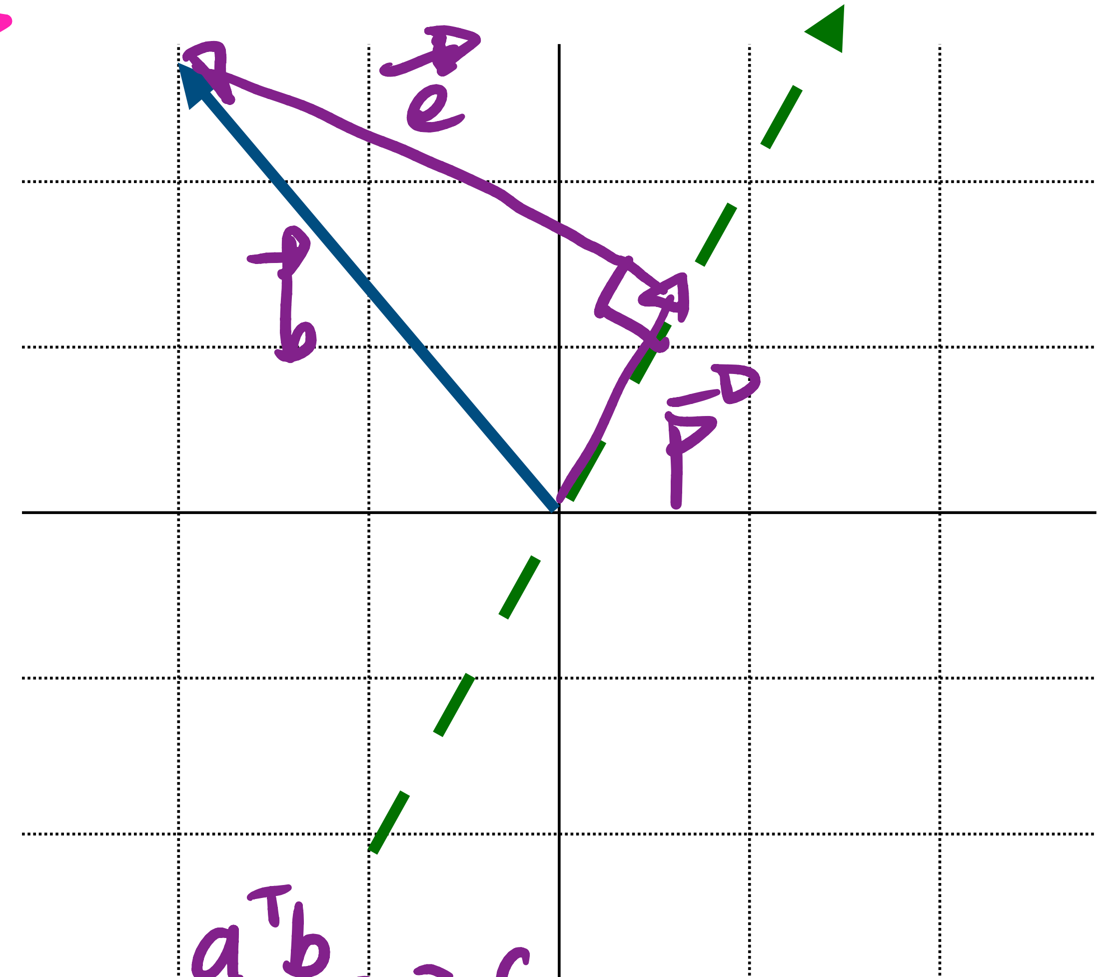
- given any new vector $\overrightarrow{b}$, what is the closest point in the span of $\overrightarrow{a}$?
- let $\overrightarrow{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\overrightarrow{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
- $\overrightarrow{p}$ will be the scaled version of $\overrightarrow{a}$ that is the component of $\overrightarrow{b}$ in $\overrightarrow{a}$
- $\overrightarrow{e}$ will be the vector orthogonal to $\overrightarrow{a}$ that "goes to" $\overrightarrow{b}$



Projections

$\vec{a} = \begin{bmatrix} 3 \\ \xi \end{bmatrix}$ $\vec{e} = \begin{bmatrix} \xi \\ \xi \end{bmatrix}$

1. $\vec{b} = \vec{p} + \vec{e}$
2. $\vec{p} = c * \vec{a}$ → scalar, we want to know its value
3. $\vec{e} = \vec{b} - \vec{p}$



- $\vec{a}$ and $\vec{e}$ are orthogonal

$a^T e = 0$ (dot prod)

- substitute #3 into + replace $\vec{p}$ w/ #2

$a^T(\vec{b} - c\vec{a}) = 0$ $\quad a^T b - a^T c a = 0$

$a^T b = c a^T a$ → $\dfrac{a^T b}{a^T a} = c$

$$\frac{a^T b}{a^T a} = c \qquad a = (2 \times 1) = \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} \qquad b = (2 \times 1)$$

$$a = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad a^T = \begin{bmatrix} 3 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\frac{(1 \times 2)(2 \times 1)}{(1 \times 2)(2 \times 1)} = \frac{\text{scalar}}{\text{scalar}}$$

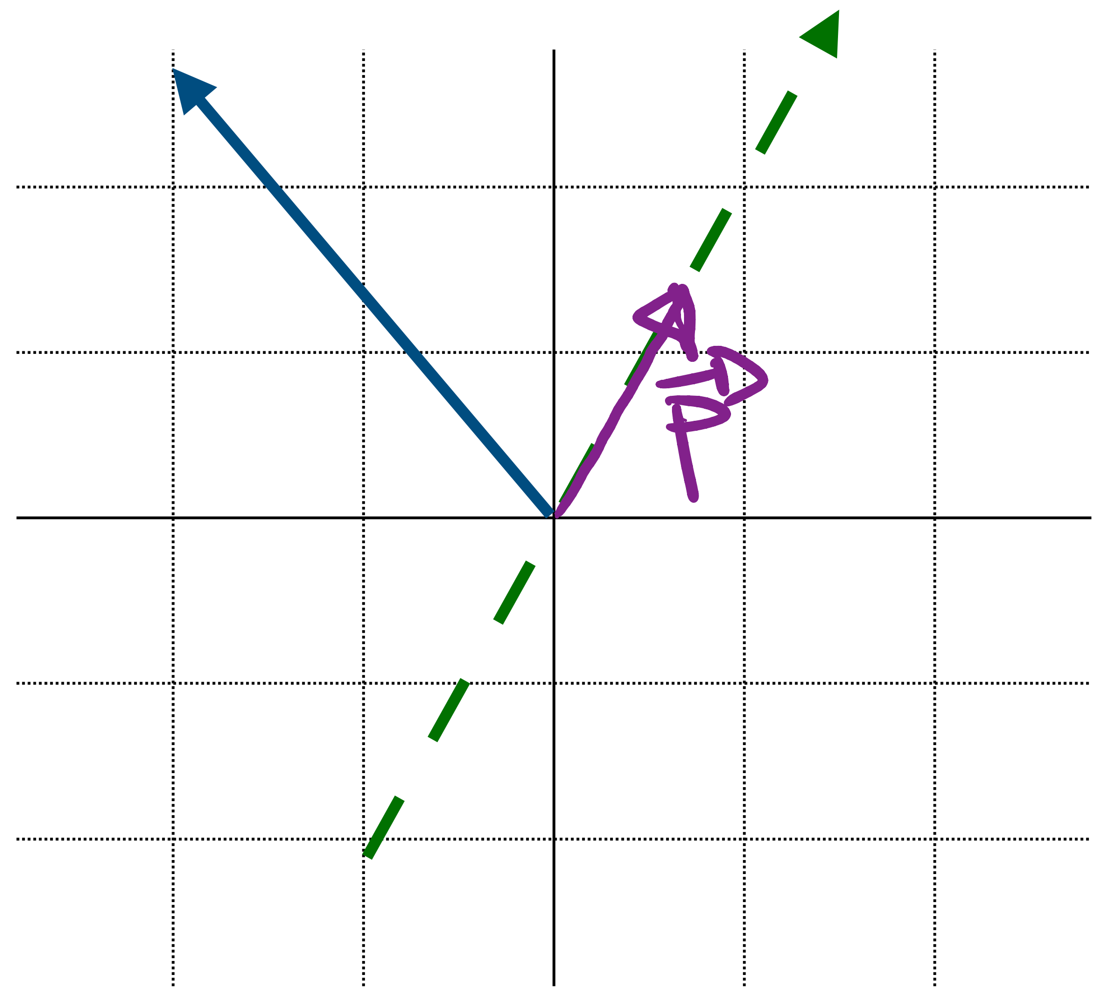
$$\frac{[3 \;\; 4]}{[3 \;\; 4]} \qquad \frac{\cancel{X}\, Y}{\cancel{X}\, Z}$$

→ you can't just eliminate these

Projections onto a line (example)

- let $\vec{a} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$
- $\vec{b} = \vec{p} + \vec{e}$
- $\boxed{\vec{p} = c * \vec{a}}$
- $c = \dfrac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}$

$$\vec{p} = \frac{\begin{bmatrix} 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}}{a^T a} \vec{a} = \frac{8}{20} \vec{a} = \begin{bmatrix} 4/5 \\ 8/5 \end{bmatrix}$$



Projections onto a plane

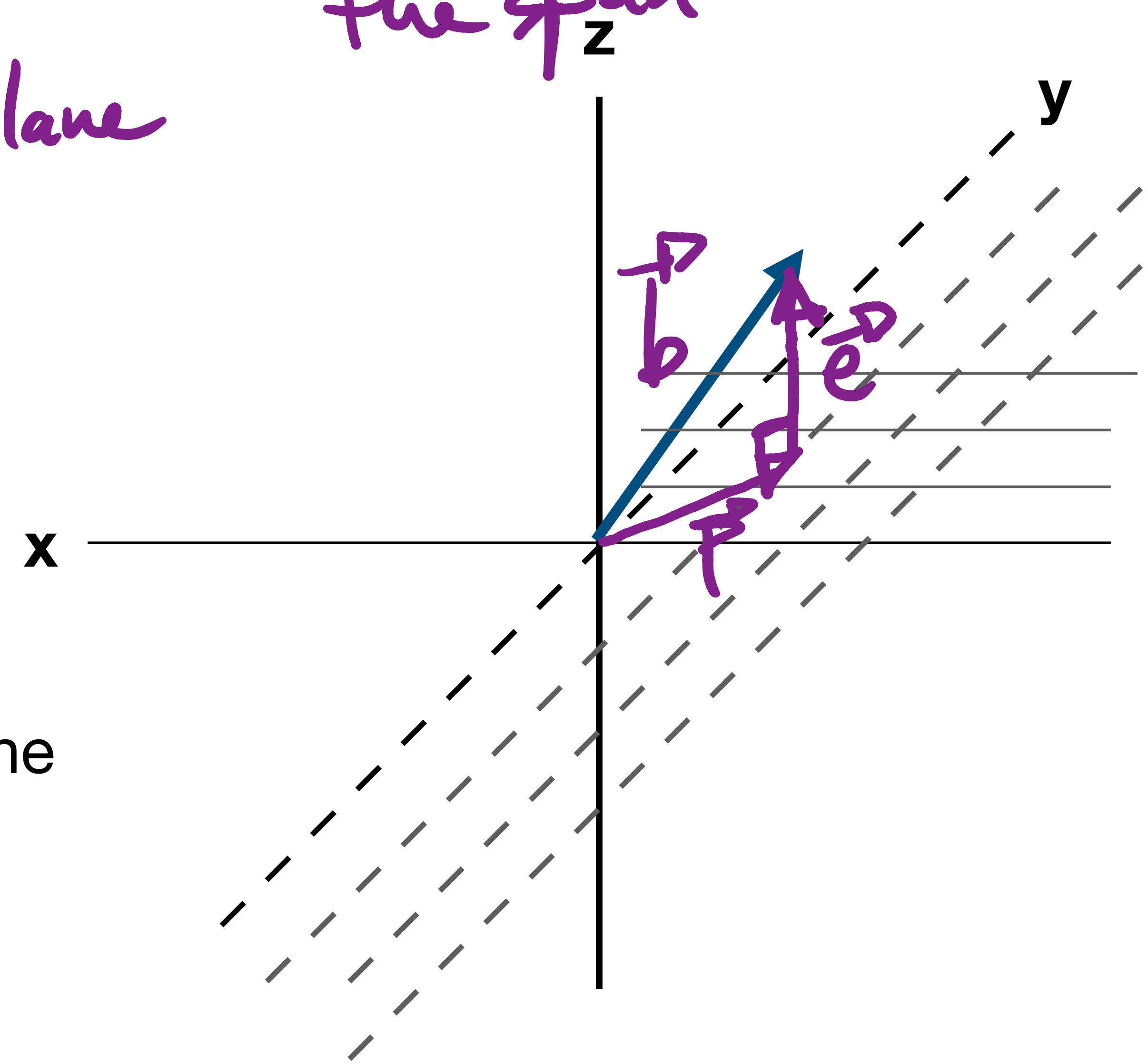
$A\vec{x} \neq \underline{\vec{b}}$

not in the span

- Now, suppose that we are projecting onto the span of $\left\{ \underbrace{\begin{bmatrix}0\\3\\0\end{bmatrix}}_{a_0}, \underbrace{\begin{bmatrix}2\\1\\0\end{bmatrix}}_{a_1} \right\}$ xy plane

- $\vec{b} = \begin{bmatrix}3\\1\\5\end{bmatrix}$ and we'd like to project it onto this span

- Goal: find $\vec{p}$ closest to $\vec{b}$ with $\vec{p}$ in the span of $\vec{a}_0$ and $\vec{a}_1$



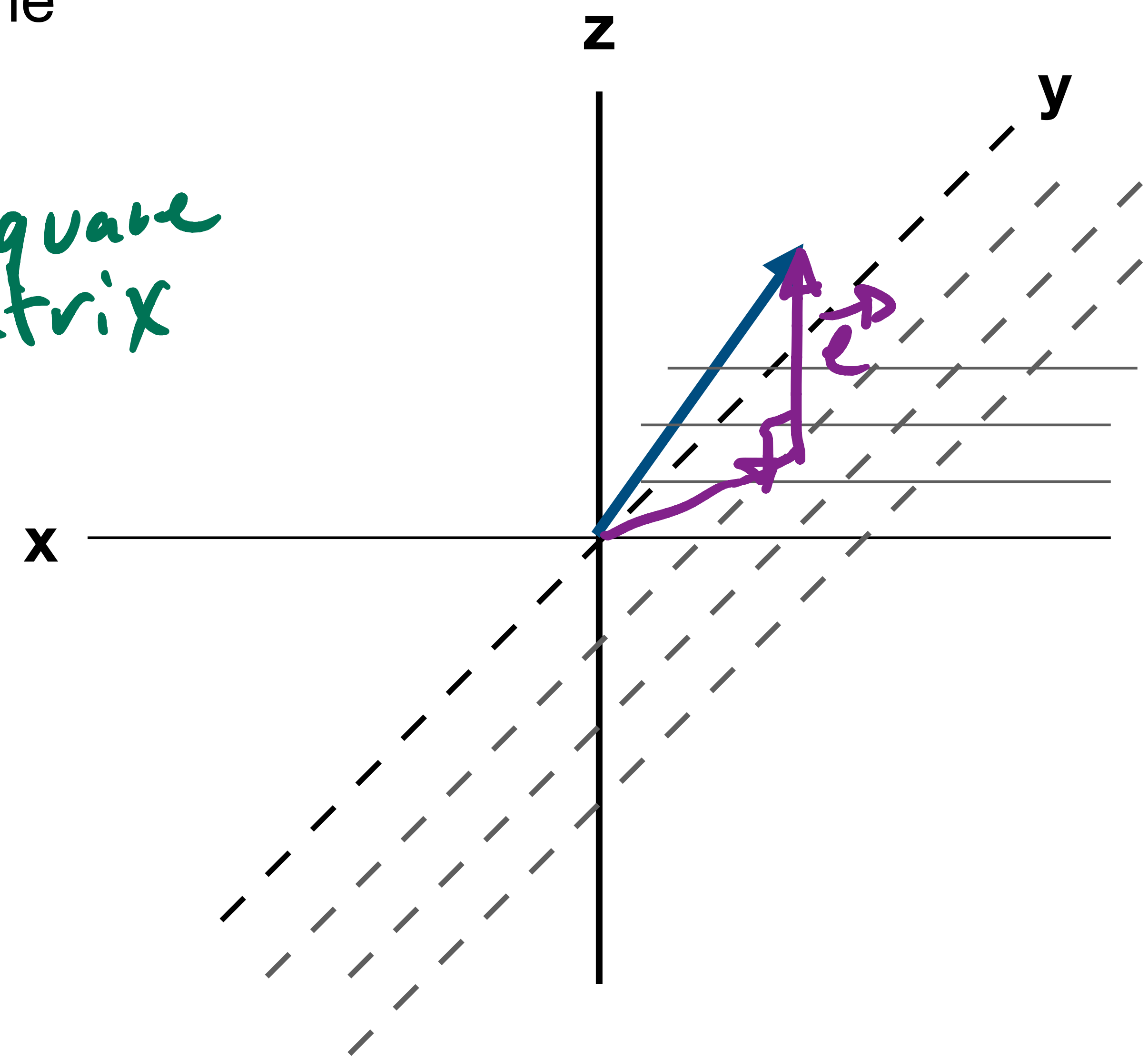
Projections onto a plane

- Goal: find $\vec{p}$ closest to $\vec{b}$ with $\vec{p}$ in the span of $\vec{a}_0$ and $\vec{a}_1$

a square matrix

1. $\vec{b} = \vec{p} + \vec{e}$
2. $\underline{\vec{p} = x_0a_0 + x_1a_1} = A\underline{x}$
3. $\vec{e} = \vec{b} - \underline{A\underline{x}}$

→ we need to solve for x



Projections onto a plane

1. $\overrightarrow{b} = \overrightarrow{p} + \overrightarrow{e}$

2. $\overrightarrow{p} = x_0 a_0 + x_1 a_1 = Ax$

3. $\overrightarrow{e} = \overrightarrow{b} - Ax$

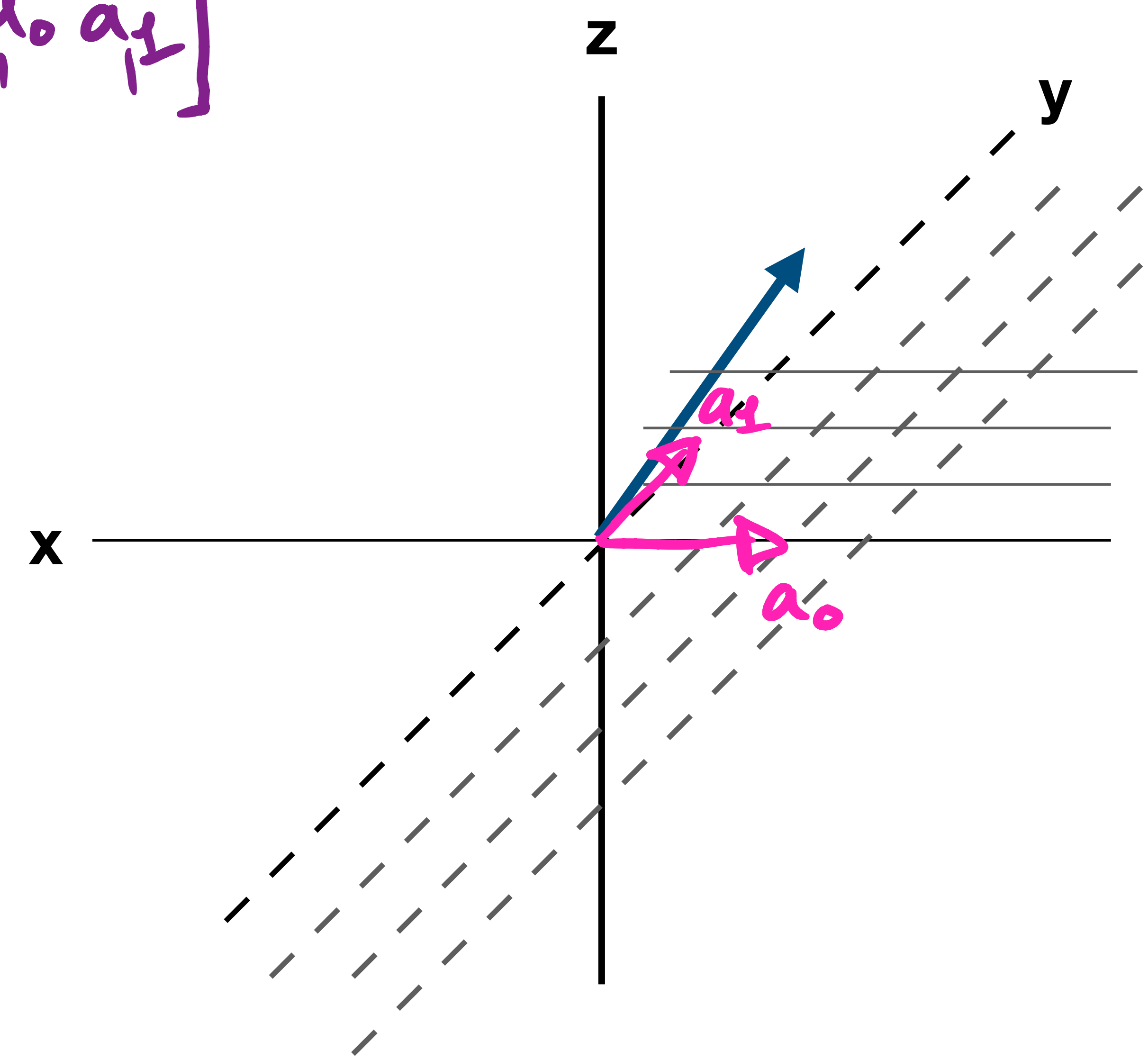
$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$

$\vec{e}$ is orthogonal to A so:

$\boxed{a_0^T (b - Ax)} = 0$ → dot prod of a basis of A w/ $\vec{e}$

$a_1^T (b - Ax) = 0$

$\begin{bmatrix} — a_0 — \\ — a_1 — \end{bmatrix} (b - Ax) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → A^T to flip



Projections onto a plane

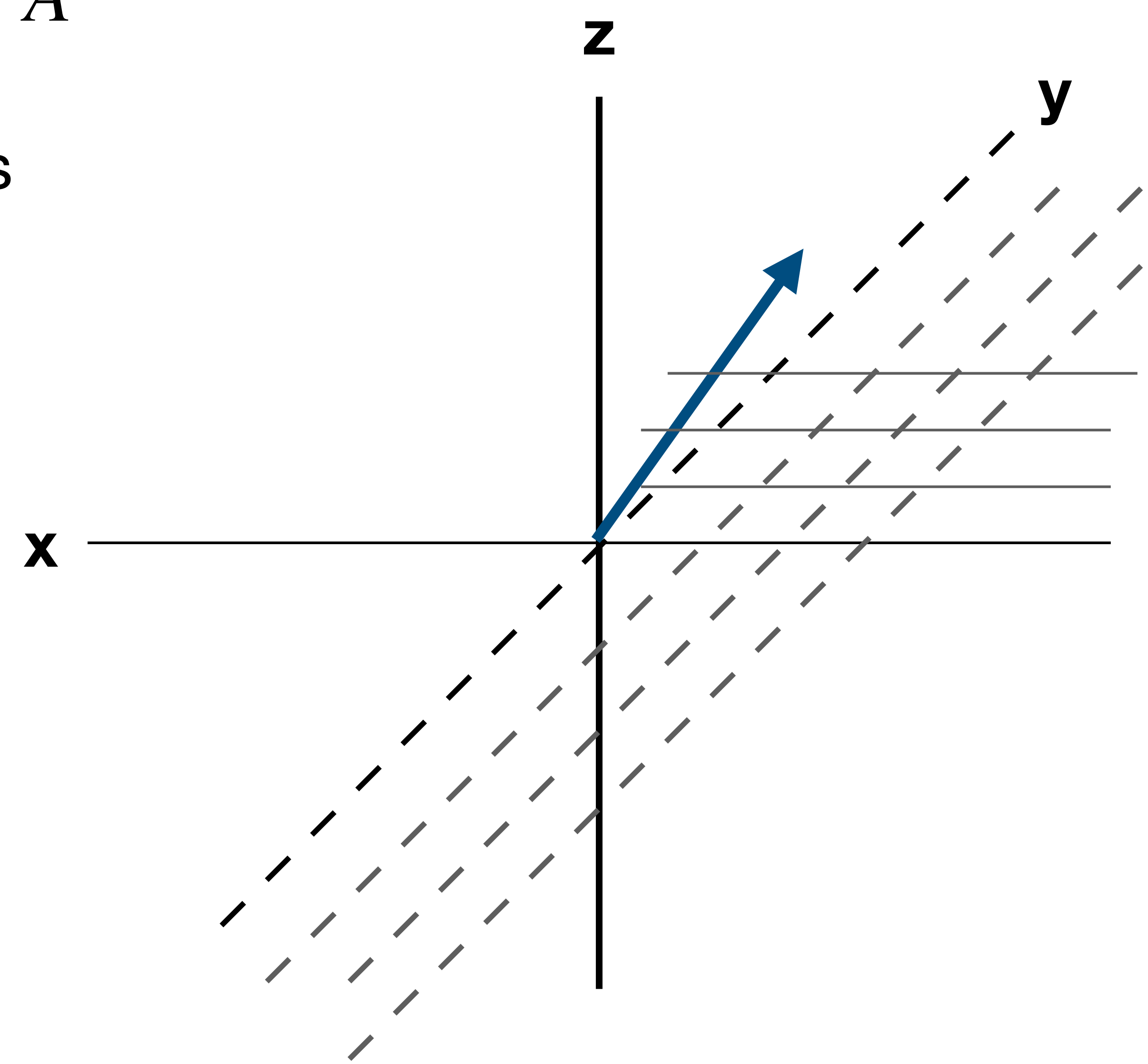
- This leaves us with the components of A as row vectors, but if we take the transpose (A^T), we get column vectors again

$$A^T(b - Ax) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T b - A^T A x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T b = A^T A x$$ → we want x

⚠ Stuck !!



Matrix Inverses

* for square matrices

- We have a notion of complementary matrices that "cancel" each other out
- But what does it mean to "cancel out" a matrix?
- What we'd like is: ?? * $Cx = x$

↳ some matrix w/ the same shape as C

- So what matrix can we multiply with x that results in x?

$$\boxed{??}\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$$

↳ some <u>other</u> matrix

↳ $C^{-1}C$ to be this

Matrix Inverses

- The **inverse** of a square matrix is the matrix that we can multiply it by to result in the **identity matrix**

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $C^{-1}C = I$

- Which gives us $C^{-1}Cx = Ix = x$

Projections onto a plane

A is matrix that is the vectors defining the span that we are projecting into

- So we had the equation $A^Tb = A^TAx$ and we are trying to solve for x

↳ mult. both sides by $(A^TA)^{-1}$

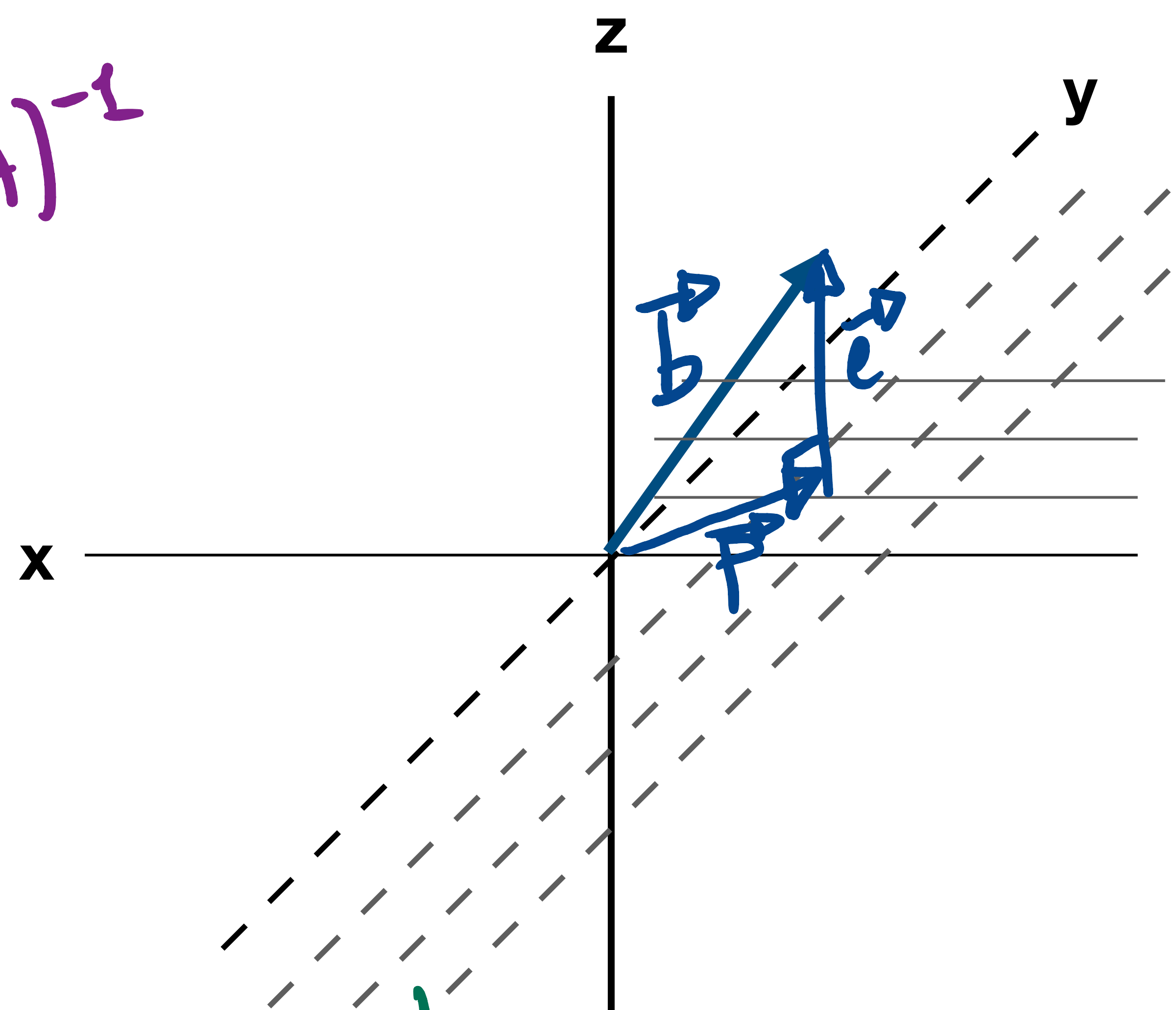
$$(A^TA)^{-1}A^Tb = \cancel{(A^TA)^{-1}A^TA}\,x$$

$$(A^TA)^{-1}A^Tb = x$$

- Finally, we can compute the projection $\overrightarrow{p}$ by substituting in x to our original observation that $\overrightarrow{p} = Ax$

$$\vec{p} = A(A^TA)^{-1}A^Tb$$

↳ do these comp. w/ your computer



x is the linear combination of the vectors in A that get you to $\vec{p}$

Projections onto a plane

$A = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix}$

ICA Question 1: what is $\overrightarrow{p}$ when $\overrightarrow{b} \in span(\{a_0, a_1\})$?

$\vec{b} = x_0 a_0 + x_1 a_1$

→ a line, a 2d plane

First, draw a picture then answer based on the picture. Then, do the math to solve for $\overrightarrow{p}$.

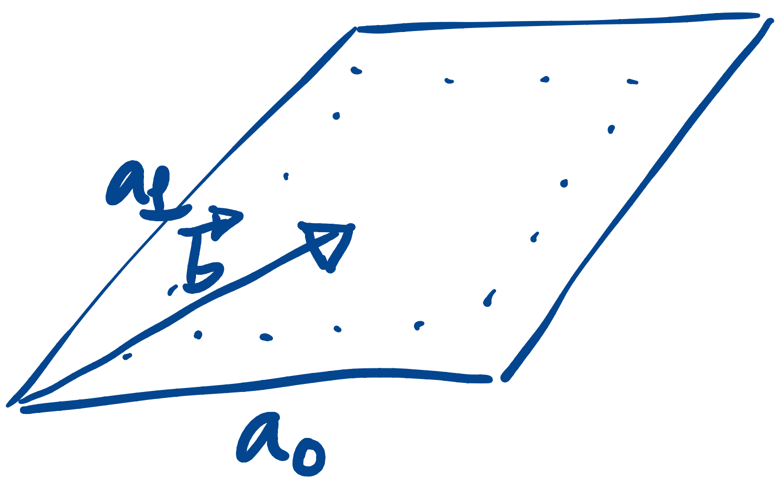
Remember: 1) $\vec{p} = Ax$

2) $\vec{p} = A(A^T A)^{-1} A^T b$

sub #1 into #2 following #3

↓

see next slide!



$a_0 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ $a_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

3) $\vec{p} = \vec{b}$

We Know

$\vec{b} = \begin{bmatrix} | & | \\ a_0 & a_1 \\ | & | \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = Ax$ because we have defined it as in the span of A

$\vec{p} = A(A^TA)^{-1}A^T\vec{b}$ from our derivation (slide 16)

$= A(A^TA)^{-1}A^TAx$ substituting in b

$\vec{p} = Ax$

Projections onto a plane

ICA Question 2: what is $\vec{p}$ when $\vec{b}$ is orthogonal to all a_i?

↳ all vectors that define the span

First, draw a picture then answer based on the picture. Then, do the math to solve for $\vec{p}$ to verify your answer.

span of a

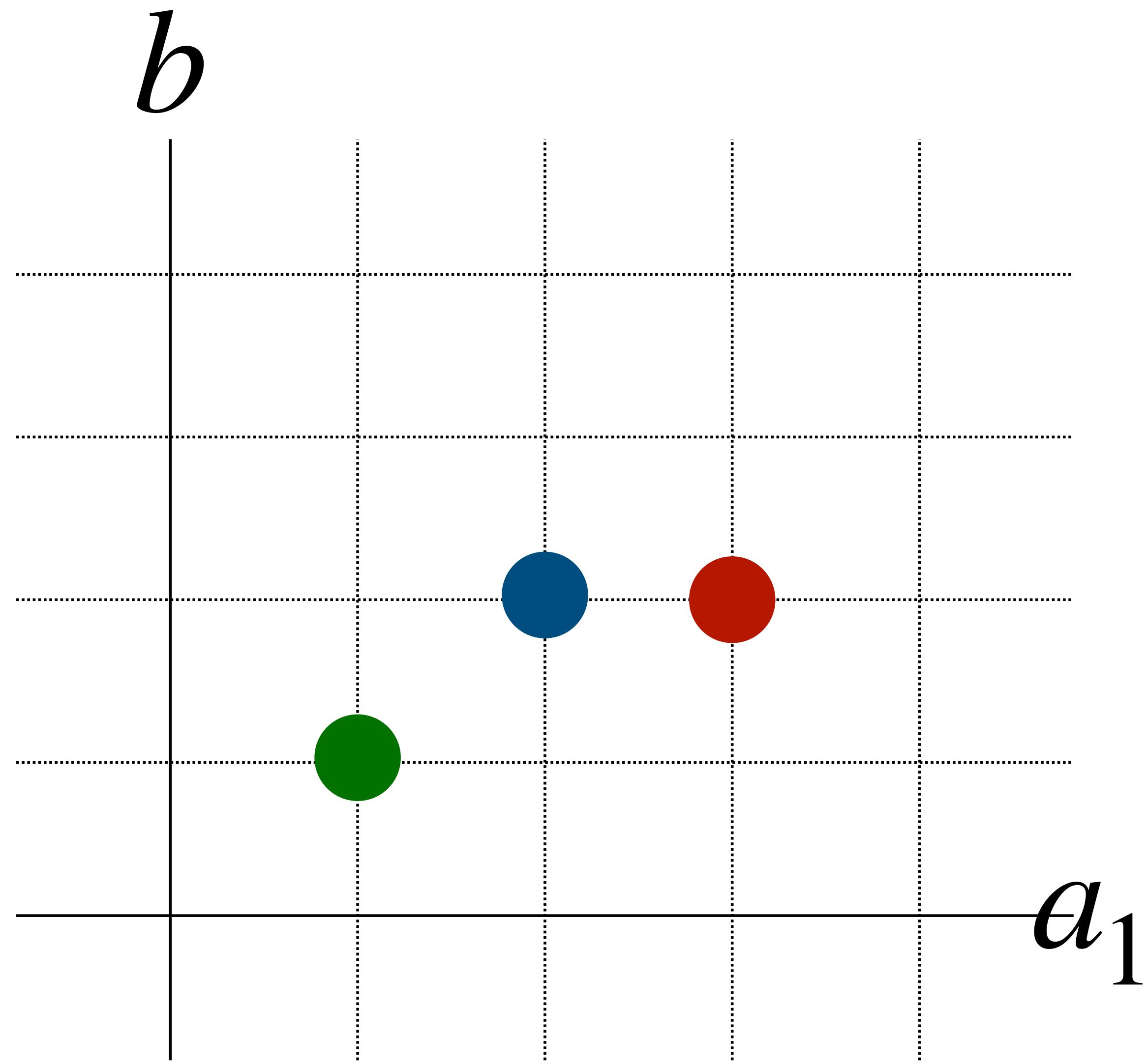
$\vec{b} = \vec{p} + \vec{e}$

↳ component of $\vec{b}$ in $\vec{a}$ → 0

Line of best fit

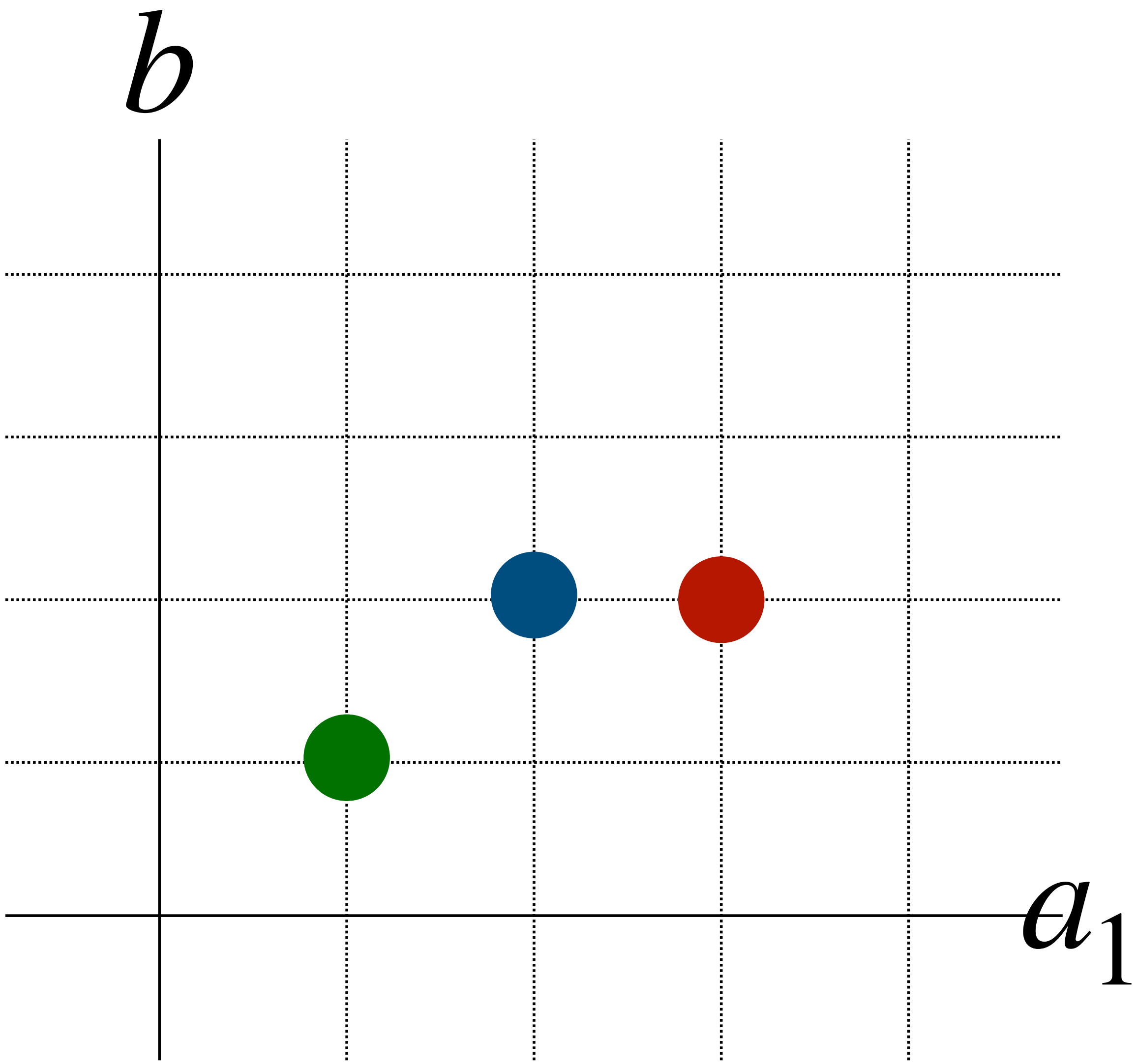
we'll do this on Monday!

- We'll want to allow for lines that do not go through the origin
- Recall the trick that we used with perceptrons: add a bias term
- When solving for the line of best fit, we'll do the same thing!

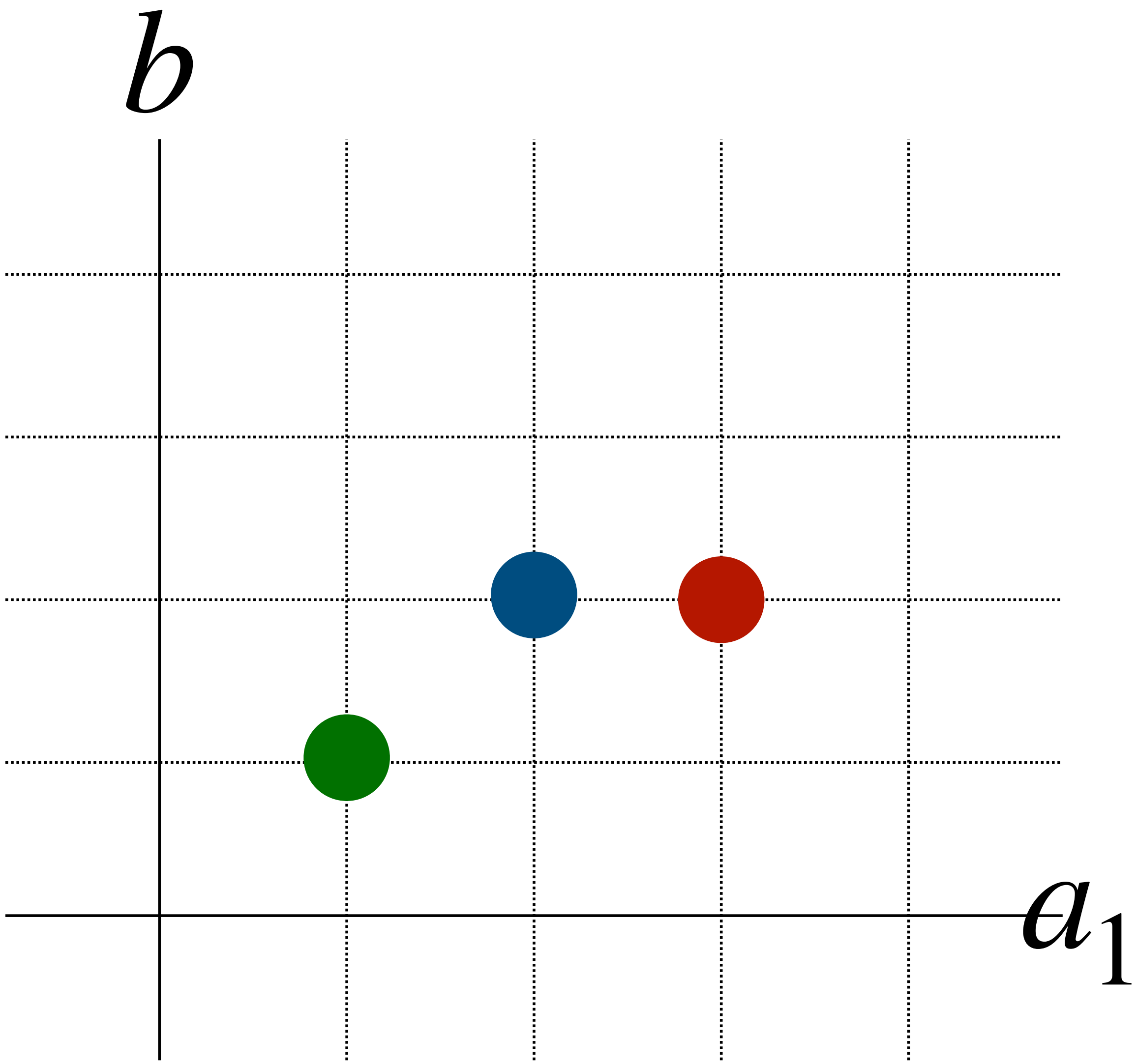


Line of best fit

- When we have a system where $Ax = b$ has no solutions
- (There is no linear combination of x in the span of A that makes b)
- Model of a line: $x_0 + x_1 a_1 = b$

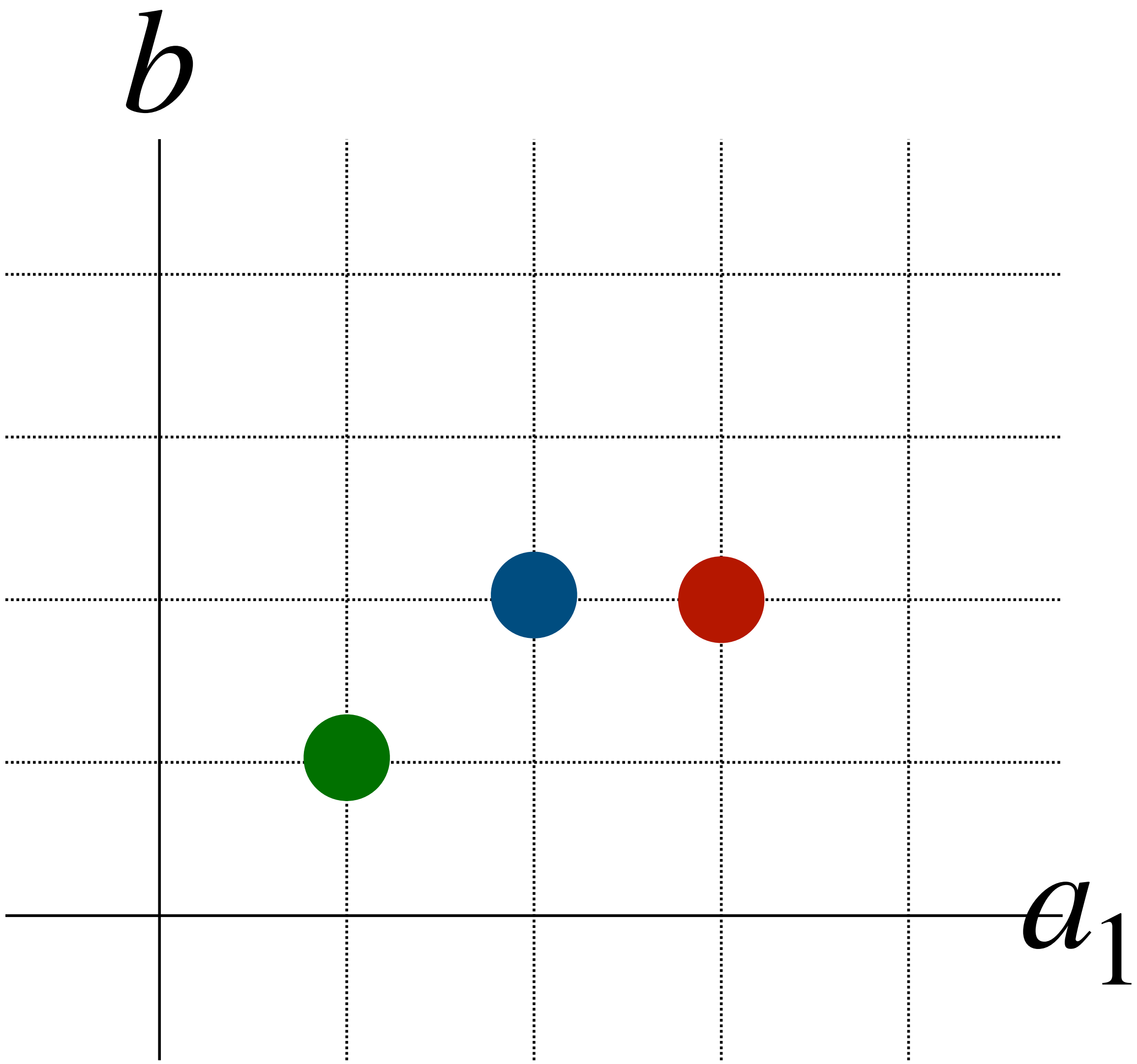


Line of best fit

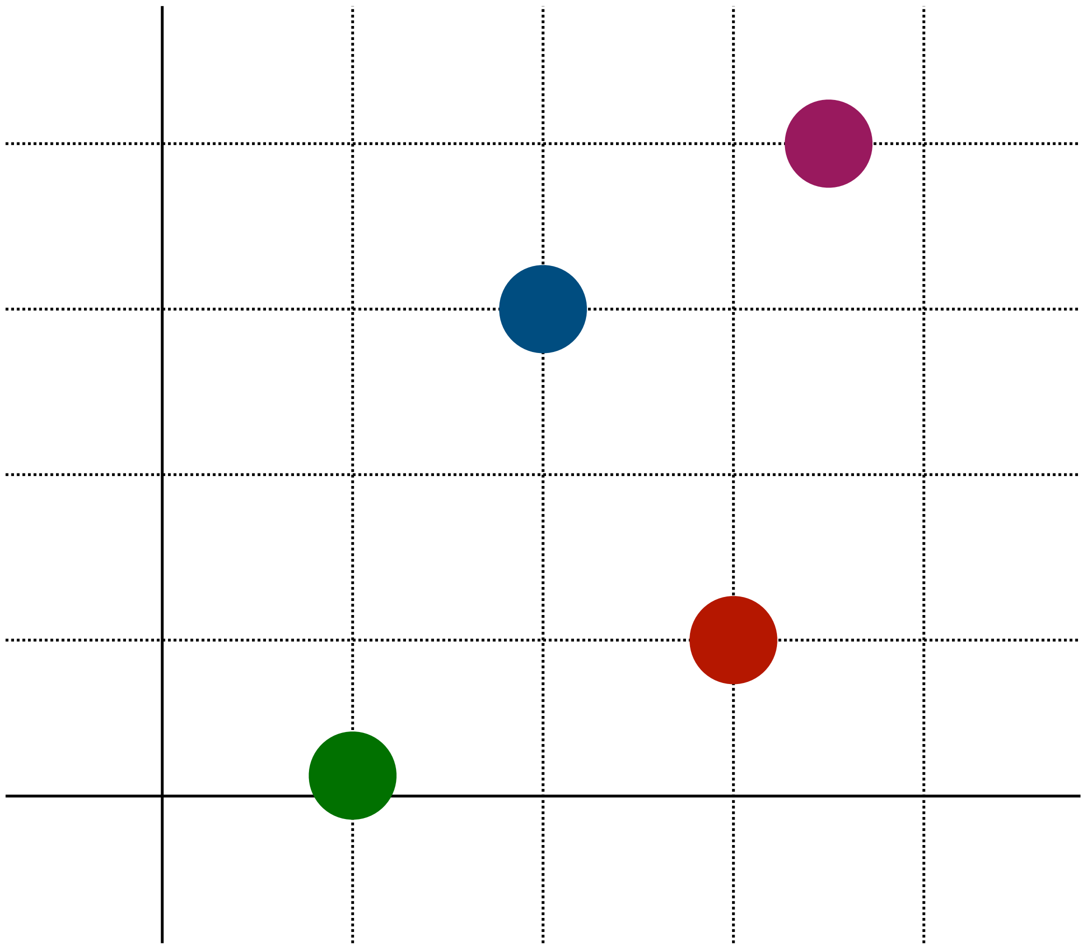


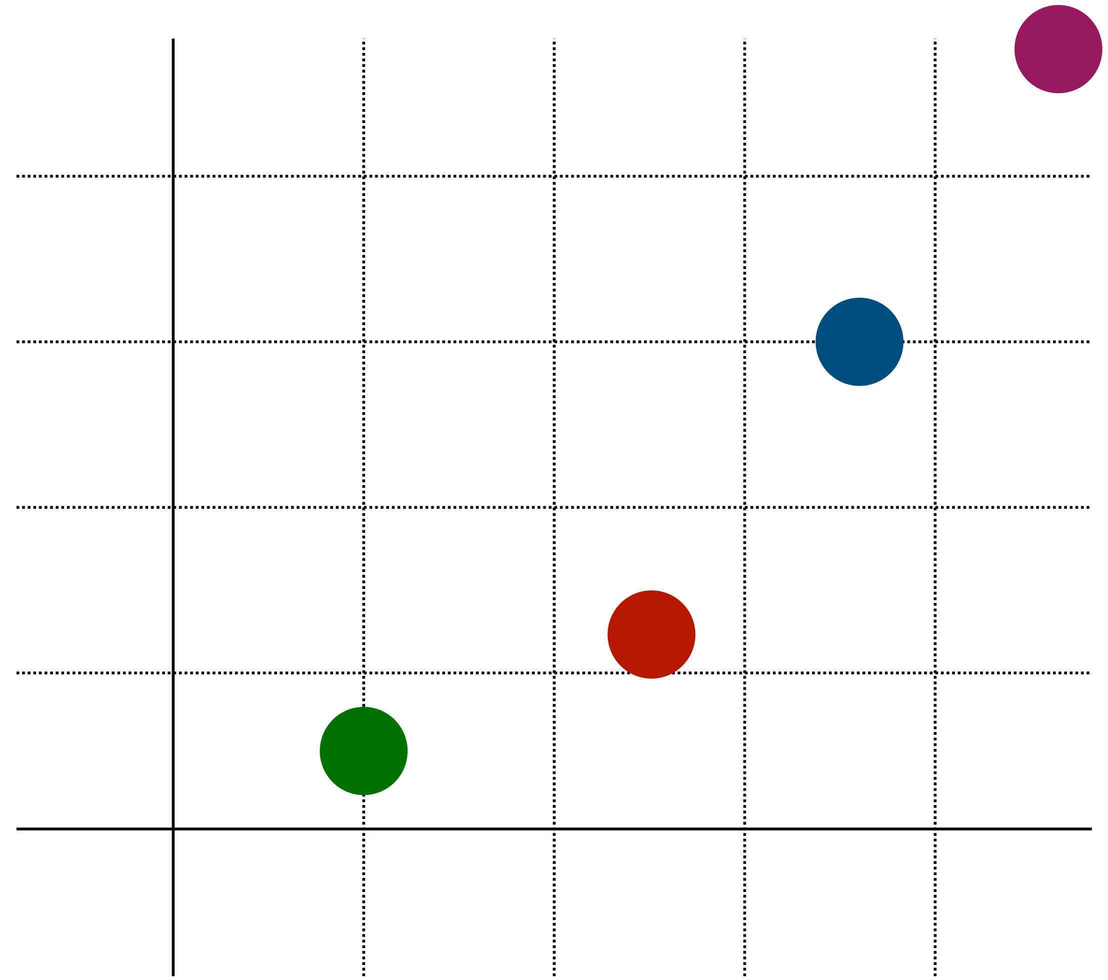
Line of best fit

- When we have a system where $Ax = b$ has no solutions
- (There is no linear combination of x in the span of A that makes b)
- We'll solve $Ax = p$ instead where p is the projection of b into the columns of A



Next time: fitting polynomial lines





python and matrix math

- Python is a programming language!
- It has some libraries that make doing matrix math quite convenient (and fast!)
- You all saw python when we looked at the perceptron examples...
- Now we'll take a look at doing some matrix manipulations together

python and matrix math

In [1]:
```
import numpy as np
```

In [2]:
```
# building a matrix
A = np.array([[1, 2, 3], [4, 5, 6]])
# asking for the dimensions of the matrix
A.shape
```

Out[2]: (2, 3)

In [3]:
```
# building a vector
x = np.array([[1, 2, 3]])
x.shape
```

Out[3]: (1, 3)

In [6]:
```
# getting the transpose of a matrix/vector
print(A.T)
print(A.T.shape)
print(x.T)
print(x.T.shape)
```

```
[[1 4]
 [2 5]
 [3 6]]
(3, 2)
[[1]
 [2]
 [3]]
(3, 1)
```

python and matrix math

In [8]:
```
# doing some matrix multiplies
A @ x  # won't work, shapes don't match!
```

```
---------------------------------------------------------------------------
ValueError                                Traceback (most recent call last)
<ipython-input-8-23a2bc3ce7ea> in <module>
      1 # doing some matrix multiplies
----> 2 A @ x  # won't work, shapes don't match!

ValueError: matmul: Input operand 1 has a mismatch in its core dimension 0, with gufunc sig
nature (n?,k),(k,m?)->(n?,m?) (size 1 is different from 3)
```

In [9]:
```
A @ x.T
```

Out[9]:
```
array([[14],
       [32]])
```

In [12]:
```
# get the inverse of a matrix
C = np.array([[1, 2], [4, 5]])
C_inv = np.linalg.inv(C)
print(C_inv)
```

```
[[-1.66666667  0.66666667]
 [ 1.33333333 -0.33333333]]
```

Schedule

Turn in ICA 8 on Gradescope

HW 2 is due on Sunday

Quiz-test 1 is in class next Thursday. → Send me an email now if you have any schedule concerns

Mon	Tue	Wed	Thu	Fri	Sat	Sun
February 7th Lecture 7 - Vector spaces **in Snell Engineering 108**	**Felix OH Calendly**		Lecture 8 - line of best fit **Felix OH Calendly**			**HW 2 due @ 11:59pm**
February 14th Lecture 9 - Polynomial best fit	**Felix OH Calendly**		Lecture 10 - **QUIZ 1 (HW 1 - 2), in class** **Felix OH Calendly**			