Logic and Computation – CS 2800
Fall 2019

Lecture 13
Boolean formula simplification
Equational proofs

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Outline

- Properties of Boolean operators
- Simplification of Boolean formulas
- Equational proofs

Quiz

- If we have the Boolean implication operator $\rightarrow$, and the Boolean constants $true, false$ (t, nil), then we can implement the Boolean negation operator $\neg$
  - Note: you can use implication, true, and false, as many times as you want

A. Yes

B. No

Quiz

- If we have the Boolean implication operator $\rightarrow$, and the Boolean constants $true, false$ (t, nil), then we can implement the Boolean disjunction operator $\vee$
  - Note: you can use implication, true, and false, as many times as you want

A. Yes

B. No

Boolean formula simplification

Equivalent formulas

- What does it mean for two Boolean formulas ϕ_1 and ϕ_2 to be **equivalent**?
  - The formula $\phi_1 \equiv \phi_2$ is valid
  - The formula ϕ_1 implies (or "is stronger than") ϕ_2 (meaning that whenever ϕ_1 is true, ϕ_2 is also true), but also ϕ_2 implies ϕ_1
  - The formulas have identical last columns in their truth tables
  - The formulas represent the same Boolean function
  - Different syntax, same semantics!

Quiz

- Consider the three formulas: (1) p, (2) $p \wedge q$, (3) $p \vee q$
- Order them w.r.t. their "strength":
  A. (1) stronger than (2) and (2) stronger than (3)
  B. (2) stronger than (1) and (1) stronger than (3)
  C. (2) stronger than (3) and (3) stronger than (1)
  D. (3) stronger than (2) and (2) stronger than (1)
  E. None of the above

Properties of Boolean operators

- Review lecture notes, section 3.3

Quiz

- What does the formula $\neg((p \oplus p) \Rightarrow q)$ simplify to?
  A. $p \Rightarrow q$
  B. $q \Rightarrow p$
  C. $\neg q \Rightarrow \neg p$
  D. true
  E. false
  F. None of the above

Another example

- Let's try to simplify this: $\neg(p \oplus (p \Rightarrow q))$

$$\neg\big(p \oplus (p \Rightarrow q)\big)$$
$$= \{\neg(\phi_1 \oplus \phi_2) \equiv (\phi_1 \equiv \phi_2)\}$$
$$p \equiv (p \Rightarrow q)$$
= { Shannon }
$$\Big(p \wedge \big(1 \equiv (1 \Rightarrow q)\big)\Big) \vee \Big(\neg p \wedge \big(0 \equiv (0 \Rightarrow q)\big)\Big)$$
$$= \{(1 \Rightarrow \phi) \equiv \phi, (0 \Rightarrow \phi) \equiv 1\}$$
$$\big(p \wedge (1 \equiv q)\big) \vee \big(\neg p \wedge (0 \equiv 1)\big)$$
$$= \{(1 \equiv \phi) \equiv \phi, (0 \equiv 1) \equiv 0\}$$
$$(p \wedge q) \vee (\neg p \wedge 0)$$
$$= \{(\phi \wedge 0) \equiv 0\}$$
$$(p \wedge q) \vee 0$$
$$= \{(\phi \vee 0) \equiv \phi\}$$
$$p \wedge q$$

Another example

- Let's try to simplify this: $p \land (p \lor q)$

Also called "reasoning by cases"

$$p \wedge (p \vee q)$$

= { Shannon }

$$\Big(p \wedge \big(1 \wedge (1 \vee q)\big)\Big) \vee \Big(\neg p \wedge \big(0 \wedge (0 \vee q)\big)\Big)$$

$$= \{(1 \vee \phi) \equiv 1, (0 \wedge \phi) \equiv 0\}$$

$$(p \wedge 1) \vee (\neg p \wedge 0)$$

$$= \{(1 \wedge \phi) \equiv \phi, (0 \wedge \phi) \equiv 0\}$$

$$p \vee 0$$

$$= \{(\phi \vee 0) \equiv \phi\}$$

$$p$$

Equational proofs

Equational proofs

- Proofs that have the form:

$$\begin{array}{c} \phi \\ = \{\text{ justification 1}\} \\ \phi_1 \\ = \{\text{ justification 2}\} \\ \phi_2 \\ = \{\text{ justification 3}\} \\ \dots \\ = \{\text{ justification N}\} \\ \psi \end{array}$$

- This proves that the formula $\phi \equiv \psi$ is valid, i.e., that the formulas ϕ and ψ are equivalent

- If ψ is the formula “false”, this proves that ϕ is ???
- If ψ is the formula “true”, this proves that ϕ is ???

Equational proofs

- This proves that the formula $\phi \equiv \psi$ is valid, i.e., that the formulas ϕ and ψ are equivalent
- If ψ is the formula "false", this proves that ϕ is unsatisfiable
- If ψ is the formula "true", this proves that ϕ is valid
- Justifications should have enough details to convince the reader!

$$\phi$$
= { justification 1}
$$\phi_1$$
= { justification 2}
$$\phi_2$$
= { justification 3}

…

= { justification N}
$$\psi$$

= vs ≡

- Should we use = or ≡ ?
- If the formulas are Boolean we can use ≡

- But we will use the same formal next in equational reasoning, where formulas can be non-Boolean, e.g., $x + 0 = x$

- No worries if you use = everywhere

$$
\begin{array}{c}
\phi \\
\equiv \{\text{ justification 1}\} \\
\phi_1 \\
\equiv \{\text{ justification 2}\} \\
\phi_2 \\
\equiv \{\text{ justification 3}\} \\
\ldots \\
\equiv \{\text{ justification N}\} \\
\psi
\end{array}
$$

Equational proofs

- This proves that the formulas ϕ and ψ are equivalent
- Why is this a valid proof?
- Because of transitivity of = (and also of $\equiv$):

$$\phi_1 \equiv \phi_2 \text{ and } \phi_2 \equiv \phi_3$$
$$\text{imply } \phi_1 \equiv \phi_3$$

ϕ
= { justification 1}
ϕ_1
= { justification 2}
ϕ_2
= { justification 3}
…
= { justification N}
ψ

Equational proofs

- Instead of = or ≡, we can also use other transitive relations, e.g., ⇒ (implies)
- Transitivity of ⇒:

$$\phi_1 \Rightarrow \phi_2 \text{ and } \phi_2 \Rightarrow \phi_3$$
$$\text{imply } \phi_1 \Rightarrow \phi_3$$

- This proves $\phi \Rightarrow \psi$

$$\phi$$
$$\Rightarrow \{\text{ justification 1}\}$$
$$\phi_1$$
$$\Rightarrow \{\text{ justification 2}\}$$
$$\phi_2$$
$$\Rightarrow \{\text{ justification 3}\}$$
$$\ldots$$
$$\Rightarrow \{\text{ justification N}\}$$
$$\psi$$

Equational proofs

- We can also use other transitive relations, e.g., $\geq$

- Transitivity of $\geq$:

  $\phi_1 \geq \phi_2$ and $\phi_2 \geq \phi_3$ imply $\phi_1 \geq \phi_3$

- This proves $\phi \geq \psi$

$$
\begin{array}{c}
\phi \\
\geq \{\text{ justification 1}\} \\
\phi_1 \\
\geq \{\text{ justification 2}\} \\
\phi_2 \\
\geq \{\text{ justification 3}\} \\
\dots \\
\geq \{\text{ justification N}\} \\
\psi
\end{array}
$$

Equational proofs

- We can also mix many different transitive relations, e.g.:

$$
\begin{array}{c}
\phi \\
= \{ \text{ justification 1} \} \\
\phi_1 \\
\geq \{ \text{ justification 2} \} \\
\phi_2 \\
= \{ \text{ justification 3} \} \\
\ldots \\
\geq \{ \text{ justification N} \} \\
\psi
\end{array}
$$

- This proves $\phi \geq \psi$

Next time

- Equational reasoning