

CS1800

Fall 2025

Recitation 12 - Practice Questions for Quiz 4 & Final Exam

December 3 & 4, 2025

Recitations

CS1802 Recitations are dedicated time set aside to work on practice problems that specifically prepare you for the current homework or upcoming quiz.

Recitations are in-person and attendance is expected.

The solutions are published at the same time as the problems, so you can check your work. There is no need to submit anything.

Approaching the Problems

These practice problems are labelled according to which Homework or Quiz topic they will help you prepare for. You do not need to complete every practice question; we encourage you to do at least one per topic, and to prioritize the topics you would like to practice.

Instructors & Teaching Assistants

Your recitation is led by a Khoury College professor, assisted by a knowledgeable and wonderful Teaching Assistant. Professors and TAs are fantastic resources, and you have the opportunity in recitation to work with them in a smaller group -- I strongly recommend you take advantage of the time to review your solutions to these practice problems, ask for help on the homework, or review material from lecture.

For full credit, all four components (predicate -- provided, logic statement, base case, and inductive step) must be included and clearly labelled. Any application of the Inductive Hypothesis must be clearly noted. Take small steps in your proof, and remember that you are convincing the reader!

Practice Problems for Quiz 4 / Final Question #7 - Proof by Induction

A Exponential functions grow slower than factorial functions, e.g., $4^n < n!$. Prove by induction that the statement is true for all integers $n \geq 9$.

Predicate $P(x)$ states that $4^x < x!$

Logic Statement $\forall n \in \mathbb{Z} \ n \geq 9 \Rightarrow P(n)$

Base Case $n = 9$. $4^9 = 262144 < 9! = 362880$. Done!

Inductive Step $P(k) \Rightarrow P(k + 1)$ for arbitrary $k \geq 9$

Solution:

Inductive Hypothesis: Assume $P(k)$ holds, i.e., $4^k < k!$

We'll use the IH to show $P(k + 1)$, our goal is to show that $4^{(k+1)} < (k + 1)!$

Let's start on the left-hand side, using only true statements one at a time, and then we'll apply the inductive hypothesis when relevant. Note that we're working with an inequality, which makes our job easier!

$$\begin{aligned}
 4^{(k+1)} &= 4^k \cdot 4^1 \\
 &< k! \cdot 4 && \# \text{ by Inductive Hypothesis} \\
 &< k! \cdot (k + 1) && \# 4 \text{ must be } < k+1 \text{ because } k \text{ is at least } 9 \\
 &= (k + 1)! && \# \text{ definition of factorial}
 \end{aligned}$$

B Use mathematical induction to show the following inequality: for all integers $n \geq 3$, $2n+1 < 2^n$.

Predicate: $P(x)$ states that $2x + 1 < 2^x$

Solution

Logic Statement: $\forall n \in \mathbb{Z}, n \geq 3 \Rightarrow P(n)$

Base case:

$$\begin{aligned} P(3) \text{ states that } 2 \cdot 3 + 1 &= 7 \\ &< 8 \end{aligned}$$

Inductive Hypothesis:

Assume $P(k)$, i.e., $2k+1 < 2^k$

Goal:

We want to show that $P(k) \Rightarrow P(k + 1)$. I.e., assuming our inductive hypothesis, we want to show that $2(k+1)+1 < 2^{k+1}$

Start with what we know to be true

$$\begin{aligned} 2(k+1)+1 &= 2k + 2 + 1 && // \text{ multiply through by 2} \\ &= 2k + 1 + 2 && // \text{ rearrange} \\ &< 2^k + 2 && // \text{ by our inductive hypothesis} \\ &< 2^k + 2^k && // \text{ we claim that our statement is true for all } n > 3, \text{ so } 2 \\ &&& // \text{ is necessarily smaller than } 2^k \\ &= 2(2^k) && // \text{ add} \\ &= 2^{k+1} && // \text{ done!} \end{aligned}$$

C Use mathematical induction to show the following summation:

Predicate: $P(x)$ states that $\sum_{i=1}^x i^3 = \frac{(x^2)(x+1)^2}{4}$

Solution

Logic Statement: $\forall n \in \mathbb{Z}^+, P(n)$

Base case:

$$P(1), \text{ summation definition: } \sum_{i=1}^1 i^3 = 1^3 = 1$$

$$P(1), \text{ formula: } \frac{(1^2)(1+1)^2}{4} = 1$$

Inductive Hypothesis:

$$\text{Assume } P(k), \text{ i.e., } \sum_{i=1}^k i^3 = \frac{(k^2)(k+1)^2}{4}$$

Goal:

We want to show that $P(k) \Rightarrow P(k + 1)$. I.e., assuming our inductive hypothesis, we want to

$$\text{show that } \sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Goal side note/reminders: just in case we're rusty on polynomials:

$$(k + 1)^2 = (k^2 + 2k + 1)$$

$$(k + 2)^2 = (k^2 + 4k + 4)$$

Start with what we know to be true

$$\sum_{i=1}^{k+1} i^3 = 1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3 \quad // \text{ definition of summation}$$

$$= \frac{(k^2)(k+1)^2}{4} + (k + 1)^3 \quad // \text{ by inductive hypothesis}$$

$$= \frac{(k^2)(k+1)^2}{4} + \frac{4(k+1)^3}{4} \quad // \text{ common term } 4/4$$

$$= \frac{(k^2)(k+1)^2 + 4(k+1)^3}{4} \quad // \text{ merge numerator}$$

$$= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \quad // \text{ pull out } (k+1)^2 \text{ term}$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \quad // \text{ multiply the 4 through}$$

$$= \frac{(k+1)^2(k+2)^2}{4} \quad // \text{ polynomial. done!}$$

Practice Problems for Final Question #1 (Logical Equivalence)

See also: Recitation 4 Problem 1

Consider the logic statements below:

P = The Red Line is delayed

Q = Laney commutes by bike

R = Boston traffic is crazy

- A** Express the following English statement in logic, using only the statements defined above and logical operators \vee , \wedge , \neg , and/or \Rightarrow

If the Red Line is delayed or Boston traffic is crazy, then Laney commutes by bike.

Solution: $(P \vee R) \Rightarrow Q$

- B** Express the following English statement in logic, using only the statements defined above and logical operators \vee , \wedge , \neg , and/or \Rightarrow

Boston traffic isn't crazy, but Laney commutes by bike if the red line is delayed.

Solution: $\neg R \wedge (P \Rightarrow Q)$

- C** Are the expressions from Part A and Part B logically equivalent?

- **If yes...** Apply the laws of logical equivalence to prove that they are the same. Take one step at a time and label each step with one law.
- **If no...** give values for p , q and r that would yield a counterexample. Simplify both expressions to demonstrate that they are not the same.

Solution: they are not equivalent

Proof with counterexample -- option 1

Let $p = T$, $q = T$, $r = T$

Simply both expressions with these values plugged in:

$(P \vee R) \Rightarrow Q$	$\neg R \wedge (P \Rightarrow Q)$
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$(T \vee T) \Rightarrow T$ $T \Rightarrow T$ T	$\neg T \wedge (T \Rightarrow T)$ $F \wedge T$ F
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Practice Problems for Final Question #2 (Representation of Numbers and Predicate Logic)

See also: Recitation 4 Problem 2

Consider the following six-bit two's complement numbers:

- 111010_2
- 110011_2
- 010011_2
- 011011_2
- 100011_2

In the universe with only the five objects above, we define the following predicates:

- $pos(x)$... x is a positive number
- $neg(x)$... x is a negative number
- $val(x, y)$... x and y have the same absolute value
- $gt(x, y)$... x has is greater than y
- $x \neq y$... object x is different than object y

Using the predicates above, logical symbols \vee , \wedge , \neg , \Rightarrow and quantifiers \forall , \exists translate the following statements about our numbers into logic.

A There are at least two positive numbers with different absolute values.

Solution $\exists x, y pos(x) \wedge pos(y) \wedge \neg val(x, y)$

B All positive numbers have the same absolute value.

Solution $\forall x, y (pos(x) \wedge pos(y)) \Rightarrow val(x, y)$

C There is a negative number that is larger than all the other negative numbers.

Solution $\exists x \forall y neg(x) \wedge (neg(y) \wedge \neg val(x, y)) \Rightarrow gt(x, y)$

D Negate the following statement, distributing the negation fully and labelling your steps along the way:

$$\forall x \exists y (pos(x) \Rightarrow (neg(y) \wedge val(x, y)))$$

Solution

The original translates to: Every positive number has a negative number with the same absolute value. (So after we negate the predicate logic, we should end up with something like, there is a positive number for which no negative number has the same absolute value.)

Let's apply the negation.

$$\neg(\forall x \exists y (pos(x) \Rightarrow (neg(y) \wedge val(x, y))))$$

$$\exists x \forall y \neg(pos(x) \Rightarrow (neg(y) \wedge val(x, y))) \quad \text{flip the quantifiers}$$

$$\exists x \forall y \neg(\neg pos(x) \vee (neg(y) \wedge val(x, y))) \quad \text{definition of implication}$$

$$\exists x \forall y (\neg \neg pos(x) \wedge \neg(neg(y) \wedge val(x, y))) \quad \text{DeMorgan}$$

$$\exists x \forall y (pos(x) \wedge \neg(neg(y) \wedge val(x, y))) \quad \text{double negation}$$

$$\exists x \forall y (pos(x) \wedge (\neg neg(y) \vee \neg val(x, y))) \quad \text{DeMorgan}$$

$$\exists x \forall y (pos(x) \wedge (val(x, y) \Rightarrow \neg neg(y))) \quad \text{Definition of implication}$$

This translates to:

- Some positive number exists that, if it has the same absolute value as another number, that other number can't be negative.

Practice Problems for Final Question #3 (Set Equality)

See also: Recitation 7 Problem 1

Consider the following pairs of shoes owned by one of your professors, this will be the Universal Set:
{*superstar*, *endorphin pro*, *endorphin speed*, *novablast*, *samba*, *adizero adios*, *allbirds*}

From those there are three sets of interest:

- $A = \{\textit{superstar}, \textit{samba}, \textit{allbirds}\}$ casual shoes
- $B = \{\textit{superstar}, \textit{samba}, \textit{adizero adios}\}$ Adidas shoes
- $C = \{\textit{adizero adios}, \textit{endorphin pro}, \textit{endorphin speed}\}$ race shoes
- $D = \{\textit{adizero adios}, \textit{endorphin pro}, \textit{endorphin speed}, \textit{novablast}\}$ running shoes

Partition the shoes into the 4 subsets described below. Express each one as a set operation on A , B , C and/or D . Your solution should use the set operation symbols for intersection, union, complement, and difference, and no other symbols.

A Casual Adidas shoes

$$A \cap B$$

B The set containing Allbirds and Novablast but nothing else

$$\overline{B} \cap \overline{C}$$

C Running shoes that are race shoes or Adidas or both

$$D \cap (B \cup C)$$

Equivalent: $(D \cap B) \cup (D \cap C)$

D List the contents of both sets below in roster notation.

$$A - (B \cup C)$$

$$A \cap \overline{B} \cap \overline{C}$$

$$A - (B \cup C) = \{\textit{superstar}, \textit{samba}, \textit{allbirds}\} - \{\textit{superstar}, \textit{samba}, \textit{adizero adios}, \textit{novablast}\}$$
$$= \{\textit{allbirds}\}$$

$$A \cap \overline{B} \cap \overline{C}$$
$$= \{\textit{superstar}, \textit{samba}, \textit{allbirds}\} \cap \{\textit{endorphin pro}, \textit{endorphin speed}, \textit{novablast}, \textit{allbirds}\}$$

$$\cap \{superstar, samba, allbirds\}$$

$$= \{allbirds\}$$

- E** Will the two sets from Part D always be the same? I.e., is $A - (B \cup C) = A \cap \overline{B} \cap \overline{C}$ true for any arbitrary sets A, B, C ?
- **If yes...** Apply the laws of set equality to prove that they are the same. Take one step at a time and label each step with one law.
 - **If no...** give example elements for $A, B,$ and C that would yield a counterexample. Simplify both sets to demonstrate that they are not the same.

For full credit, take only step at a time, and clearly indicate which set equality law/definition is Applied.

Solution - always the same!

$$A - (B \cup C) = A \cap \overline{B} \cap \overline{C}$$

Solution:

$A - (B \cup C)$	start
$A \cap \overline{(B \cup C)}$	definition of difference
$A \cap (\overline{B} \cap \overline{C})$	DeMorgan

Practice Problems for Final Question #4 (Counting)

See also: Recitation 7 Problem 2

Laney visits JP Licks and sees that there are 6 vegan flavors.

- A** Laney chooses 2 different vegan flavors for a double scoop. How many possible flavor combinations can she choose?

Solution
 $C(6,2) = 15$

- B** If the order of the scoops matters (for example, chocolate on top of vanilla is different from vanilla on top of chocolate), how many possible double scoops are there?

Solution

$$P(6,2) = 30$$

- C** Laney chooses 2 vegan flavors for a double scoop, but now she might repeat a flavor. How many possible flavor combinations can she choose?

Solution - stars and bars

There are 2 stars (the scoops) and 5 bars (for the flavor categories)

$$C(7,2) = 21$$

- D** Laney buys 3 cones, each with a double scoop chosen as in part C. If the cones are unlabeled, how many sets of 3 cones could she buy?

Solution - more stars and bars

We know from Part C that we have 21 options per cone. The cones being unlabelled suggests that order doesn't matter. Is repetition ok? The question doesn't say, but from context there's nothing to stop getting the same cone more than once. This gives us

$$C(23,3) = 1771$$

- E** How many ways are there to arrange the letters in the word *VAMPIRE*?

Solution

There are no repeated letters, so this is a simple permutation $P(7, 7) = 7! = 5040$

- F** How many ways are there to arrange the letters in the word *ROOKIE*?

Solution - Now we have two O's, making the question a little more interesting, because the n's are indistinguishable.

We start with a similar answer as Part A, $P(6, 6) = 6!$

But this overcounts, assuming that $O_1 O_2 KEI$ is a different permutation than $O_2 O_1 KEI$ when in reality they are the same: *OOKEI*.

So we take our original answer and divide by the number of overcounted O arrangements, giving

$$\text{us } \frac{P(6,6)}{P(2,2)} = \frac{6!}{2!} = 360$$

- G** How many ways are there to arrange the letters in the word *LEVEL*?

Solution - Another level, now we have two repeated letters.

We start with a similar answer as Part A, $P(5, 5) = 5!$
But this overcounts, both for the E's and for the L's.

So we take our original answer and divide by both of our overcounted arrangements, giving us

$$\frac{P(5,5)}{P(2,2) \cdot P(2,2)} = \frac{5!}{2! \cdot 2!} = 30$$

- H** Laney is waiting for the eastbound Green Line at Hynes. Three of the four green line branches come through this station, at unpredictable intervals. If Laney takes 8 trips where 3 are on the B branch, 2 are on the C branch, and 3 are on the D branch, how many ways can we order a sequence of B, C, and D trains? (For example, CDCDBBBD is one possibility.)

Solution: $\frac{8!}{3! \cdot 2! \cdot 3!} = 560$

We again start with $P(8, 8) = 8!$, and divide out the overcounted B, C, and D's.

- I** There are 12 dogs at the dog park. How many ways can you split them into three adorable teams of equal size, labelled Team A, Team B, and Team C?

Solution: $C(12, 4) \cdot C(8, 4) \cdot C(4, 4) = 495 \cdot 70 = 34650$

- J** There are 12 dogs at the dog park. How many ways can you split them into three adorable teams of equal size without any labels?

Solution: $34650/3! = 5775$

Practice Problems for Final Question #5 (Probability)
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See also: Recitation 10 Problem 1

- A** You flip three fair coins. What is the probability that all three are heads, given that at least two of them are heads?

Solution:

This is a conditional probability, $P(E|F) = \frac{P(E \cap F)}{P(F)}$, with E = the set of all 3-flip sequences where all three are heads, and F = the set of 3-flip sequences where at least two of them are heads.

$P(E \cap F)$ = the probability of all heads AND at least two are heads.

$$P(E \cap F) = 1/8$$

$P(F)$ = the probability that least two flips are heads. This breaks down to two cases:

1. Exactly two flips are heads: 3 outcomes
2. Exactly three flips are heads: 1 outcome

$$P(F) = 4/8$$

Finally, we put them together to get $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{4/8} = 1/8 \cdot 8/4 = 1/4$

B You flip 10 fair coins. What is the expected number of tails you'd see?

Solution:

This is an indicator random variable problem. For any one flip,

$$E[X_i] = 1 \cdot 1/2 + 0 \cdot 1/2 = 1/2$$

$$\text{So } E[X] = 10 \cdot E[X_i] = 5$$

C You roll two fair six-sided dice. What is the probability that both dice show 6, given that at least one die shows 6?

This is a conditional probability, $P(E|F) = \frac{P(E \cap F)}{P(F)}$, with E = the set of all two-dice rolls where both are sixes, and F = the set of all two-dice rolls where at least one is a six.

$P(E \cap F)$ = the probability of both sixes AND at least one six.

$$P(E \cap F) = 1/36$$

$P(F)$ = the probability that least one is a six, meaning...

- (6, __): 5 options for the blank
- (__ ,6): 5 options for the blank
- (6,6): one outcome

Total: 11 outcomes

$$P(F) = 11/36$$

Finally, we put them together to get $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/36}{11/36} = 1/11$

Practice Problems for Final Question #6 (Sequences and Summations)

See also: Recitation 10 Problem 2

You deposit \$100 into a savings account every month. The account earns 2% interest per month, and interest is applied before each new deposit. You make your first deposit right after opening the account, and you do this for 6 months total.

Let a_n be the amount in your account just after the n th deposit.

- A** What are the first four terms of the sequence representing the total amount of money in your account.

Solution

$$a_1 = 100$$

$$a_2 = 100 \cdot 1.02 + 100 = 102 + 100 = 202$$

$$a_3 = 202 \cdot 1.02 + 100 = 306.04$$

$$a_4 = 306.04 \cdot 1.02 + 100 = 412.1608$$

- B** Give a recursive formula for a_n including the base case.

Solution

$$a_1 = 100$$

$$a_n = a_{n-1} \cdot 1.02 + 100$$

Let's change it up and say you deposit \$50 each month, earning 1% interest per month, for 4 months. Let b_n be the amount after the n th deposit.

- C Give a recursive formula for a_n including the base case.

Solution

$$b_1 = 50$$

$$b_n = (1.01)b_{n-1} + 50$$

- D What are the four terms of the sequence b_1, b_2, b_3, b_4 ?

Solution

$$b_1 = 50$$

$$b_2 = (50 \cdot 1.01) + 50 = 100.50$$

$$b_3 = (100.50 \cdot 1.01) + 50 = 151.505$$

$$b_4 = (151.505 \cdot 1.01) + 50 = 203.02$$

At the Tesla factory, production of Cyber Trucks has been increasing ever since Prof. Derbinsky bought one. In the first month, Tesla made one Cyber Truck; in the second month they made two; in the third month they made three, and so on.

- E Give a recursive formula for a_n including the base case.

Solution

$$a_1 = 1$$

$$a_n = a_{n-1} + 1$$

- F Give a closed-form formula for the k th term a_k .

Solution

$$a_k = k$$

- G How many cars are produced in the first year?

Solution

$$\sum_{i=1}^{12} i = \frac{12 \cdot 13}{2} = 78$$

