

# SOLUTIONS

**CS1800**

**Fall 2025**

**Recitation 11 - Practice Questions for Homework 7**

**November 19 & 20, 2025**

## **Recitations**

CS1802 Recitations are dedicated time set aside to work on practice problems that specifically prepare you for the current homework or upcoming quiz.

Recitations are in-person and attendance is expected.

The solutions are published at the same time as the problems, so you can check your work. There is no need to submit anything.

## **Approaching the Problems**

These practice problems are labelled according to which Homework or Quiz topic they will help you prepare for. You do not need to complete every practice question; we encourage you to do at least one per topic, and to prioritize the topics you would like to practice.

## **Instructors & Teaching Assistants**

Your recitation is led by a Khoury College professor, assisted by a knowledgeable and wonderful Teaching Assistant. Professors and TAs are fantastic resources, and you have the opportunity in recitation to work with them in a smaller group -- I strongly recommend you take advantage of the time to review your solutions to these practice problems, ask for help on the homework, or review material from lecture.

## SOLUTIONS

**Practice Problems for (HW7, Q1) - Summations**

A Use mathematical induction to show the following summation.

**Predicate**  $P(x)$  states that  $\sum_{i=1}^x 2i = (x)(x + 1)$

**Logic Statement:**  $\forall n \in \mathbb{Z}^+ P(n)$

*Solution*

Base case  $P(1)$

Left-hand side (summation definition):  $\sum_{i=1}^1 2i = 2$

Right-hand side (formula):  $(1)(1 + 1) = 2$

Inductive step  $P(k) \Rightarrow P(k + 1)$

Inductive Hypothesis:  $P(k)$ .  $\sum_{i=1}^k 2i = (k)(k + 1)$

Goal: show  $P(k + 1)$ , i.e.,  $\sum_{i=1}^{k+1} 2i = (k + 1)(k + 2)$

$\sum_{i=1}^{k+1} 2i = 2 + 4 + 6 + \dots + 2k + 2(k + 1)$  // summation definition

$= \sum_{i=1}^k 2i + 2(k + 1)$  // Break the sum into two parts

$= (k)(k + 1) + 2(k + 1)$  // true by inductive hypothesis!!

$= k^2 + k + 2k + 2$  // multiply through

$= k^2 + 3k + 2$  // simplify

$= (k + 1)(k + 2)$  // polynomial

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**B** Use mathematical induction to show the following summation:

**Predicate**  $P(x)$  states that  $\sum_{i=1}^x \frac{1}{i(i+1)} = \frac{x}{x+1}$

**Logic Statement:**  $\forall n \in \mathbb{Z}^+ P(n)$

*Solution*

Base case  $P(1)$

$$\text{Left-hand side: } \sum_{i=1}^1 \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$$

$$\text{Right-hand side: } \frac{x}{x+1} = \frac{1}{1+1} = 1/2$$

Inductive step  $P(k) \Rightarrow P(k + 1)$

$$\text{Inductive hypothesis: } P(k). \sum_{i=1}^k \frac{1}{i(i+1)} = \frac{k}{k+1}$$

$$\text{Goal: show } P(k + 1), \text{ i.e., } \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^k \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+1+1)} \quad // \text{ Break the sum into two parts: From 1 to } k, \text{ and } k+1$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+1+1)} \quad // \text{ By inductive hypothesis}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \quad // \text{ Small step: Just changed } k+1+1 \text{ to } k+2$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \quad // \text{ Multiply top and bottom of first term by } k+2$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)} \quad // \text{ Add the fractions}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)} \quad // \text{ Multiply } k \text{ through the first term in the numerator}$$

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$$= \frac{(k+1)^2}{(k+1)(k+2)} \quad // \text{ Polynomial math}$$

$$= \frac{(k+1)(k+1)}{(k+1)(k+2)} \quad // \text{ Express the square as a multiple}$$

$$= \frac{k+1}{k+2} \quad // \text{ Remove the common } k+1$$

Now we have

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$$

Done!

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### Practice Problems (HW7, Question 2) - Inequality

A Prove by induction that the following predicate is true for all integers  $n \geq 9$ .

**Predicate**  $P(x)$  states that  $6x + 8 < x^2$

**Logic Statement**  $\forall n \in \mathbb{Z}^+, n \geq 9 \Rightarrow P(n)$

#### Base case

$P(9)$  states that  $6 \cdot 9 + 8 \leq 9^2$

Simplifying, we have  $62 \leq 81$ , which is true.

#### Inductive step $P(k) \Rightarrow P(k + 1)$

Inductive Hypothesis  $P(k)$ , assumed true:  $6k + 8 \leq k^2$

Goal: show  $P(k + 1)$ , i.e.,  $6(k + 1) + 8 \leq (k + 1)^2$

We begin with the definition of  $P(k+1)$ , given by  $6(k + 1) + 8$ .

$$\begin{aligned} 6(k + 1) + 8 &= 6k + 6 + 8 && // \text{multiply the 6 through} \\ &= (6k + 8) + 6 && // \text{re-group terms} \\ &\leq k^2 + 6 && // \text{by inductive hypothesis} \\ &\leq k^2 + 2k + 1 && // \text{because } k \text{ is least } 9, 6 \leq 2k + 1 \\ &= (k + 1)^2 && // \text{polynomial expansion.} \end{aligned}$$

Done!

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**B** When we cover the growth of functions toward the end of the semester, we'll talk about exponential functions growing slower than factorial functions, e.g., that  $3^n < n!$ . Prove by induction that this is true for all integers  $n \geq 7$ .

**Predicate**  $P(x)$  states that  $3^x < x!$

**Logic Statement**  $\forall n \in \mathbb{Z}^+ n \geq 7 \Rightarrow P(n)$

**Solution**

**Base Case** \_\_\_\_\_

**Solution:**  $n = 7$ .  $3^7 = 2187 < 7! = 5040$ . Done!

**Inductive Step**  $P(k) \Rightarrow P(k + 1)$

**Solution:**

Inductive Hypothesis: Assume  $S(k)$  holds, i.e.,  $3^k < k!$

Our goal is to show  $P(k+1)$ , i.e.,  $3^{(k+1)} < (k + 1)!$

Let's start on the left-hand side, using only true statements one at a time, and then we'll apply the inductive hypothesis when relevant. Note that we're working with an inequality, which makes our job easier!

$$\begin{aligned} 3^{(k+1)} &= 3^k \cdot 3^1 \\ &< k! \cdot 3 && \# \text{ by Inductive Hypothesis} \\ &< k! \cdot (k + 1) && \# 3 \text{ must be } < k+1 \text{ because } k \text{ is at least } 7 \\ &= (k + 1)! && \# \text{ definition of factorial} \end{aligned}$$

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### Practice Problems for (HW7, Question 3) - Divisibility

Use mathematical induction to prove the following predicate.

**Predicate**  $P(x)$  states that  $7^x - 2^x$  is a multiple of 5.

Solution (note that I use “is a multiple of \_\_\_” language instead of “5 divides” or  $5|$  b/c we didn’t really cover divisibility)

**Logic statement**  $\forall n \in \mathbb{Z}^+, P(n)$

**Base case**

$$\begin{aligned} P(1) \quad (7^1 - 2^1) &= 5 \\ 5 &= 5 \cdot 1 \quad \text{all set!} \end{aligned}$$

**Inductive step**  $P(k) \Rightarrow P(k + 1)$

Inductive Hypothesis  $P(k)$ , assumed true:  $(7^k - 2^k)$  is a multiple of 5.

$P(k + 1)$  gives us that  $(7^{k+1} - 2^{k+1})$  is a multiple of 5. This is what we want to prove.

Start by plugging in  $k+1$  to get the expression we care about, and then do things that don’t modify the value of the expression.

$$\begin{aligned} 7^{k+1} - 2^{k+1} &= 7^k \cdot 7^1 - 2^k \cdot 2^1 && // \text{exponent law } a^b a^c = a^{b+c} \\ &= 7^k \cdot (5 + 2) - 2^k \cdot 2 && // 7=5+2, \text{ we do this because 5 is useful!} \\ &= 7^k \cdot 5 + 7^k \cdot 2 - 2^k \cdot 2 && // \text{Distribute the exponent terms} \\ &= 7^k \cdot 5 + 2(7^k - 2^k) && // \text{rearrange} \end{aligned}$$

$7^k \cdot 5$  is a multiple of 5 by definition

$7^k - 2^k$  is a multiple of 5 by inductive hypothesis

$2(7^k - 2^k)$  is double a multiple of 5....  $2 \cdot 5 \cdot \text{something}$  is also a multiple of 5

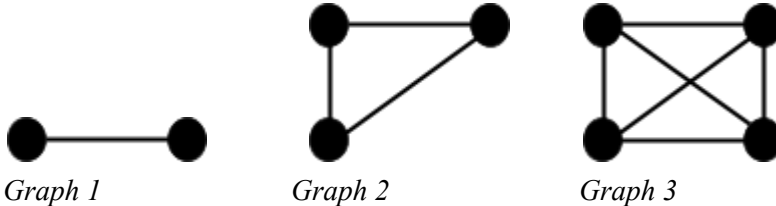
And we know that the sum of two multiples of 5 is itself a multiple of 5. We’re done!

## SOLUTIONS

### Practice Problems for (HW7, Question 4) - Data Structures

Like a set, a graph has certain properties that we can prove with mathematical induction. We'll be covering more about graphs starting on Friday, but for now all we need to know is that a graph is a collection of vertices and edges.

One type of graph is known as a *complete graph*; there is an edge between every pair of vertices. Here are three examples of a complete graphs:



Notice that...

- The complete graph with two vertices has one edge.
- The complete graph with three vertices has three edges.
- The complete graph with four vertices has six edges.

It's all part of a pattern that we can prove!

Use mathematical induction to show that the number of edges in a complete graph with  $n$  vertices is  $\frac{(n)(n-1)}{2}$ .

**Predicate:**  $P(x)$  states that a complete graph with  $n$  vertices has  $\frac{(x)(x-1)}{2}$  edges.

**Logic statement:**  $\forall n \in \mathbb{Z}^+ P(n)$

**Base case:**

$P(1)$  holds because with one vertex it must have zero edges, and  $\frac{1-0}{2} = 0$

**Solution:**

**Inductive Step:**  $P(k) \Rightarrow P(k + 1)$

*IH:* Assume it's true that  $P(k)$  holds, i.e., a graph with  $k$  vertices has  $\frac{(k)(k-1)}{2}$  edges.

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*GOAL:* We want to show that  $P(k+1)$  is true, i.e., a graph with  $k+1$  vertices has  $\frac{(k+1)(k)}{2}$  edges.

Take our graph with  $k$  vertices and  $\frac{(k)(k-1)}{2}$  edges. Add a vertex. Because it's a complete graph, the new vertex connects to all existing edges,  $k$  of them, so we've just added  $k$  new edges.

From here it's just a few simple algebraic steps. The total number of edges in our graph with now  $k+1$  vertices is...

$$\begin{aligned} & \frac{(k)(k-1)}{2} + k \\ = & \frac{(k)(k-1)}{2} + \frac{2k}{2} \\ = & \frac{k^2 - k + 2k}{2} \\ = & \frac{k^2 + k}{2} \\ = & \frac{(k+1)(k)}{2} \end{aligned}$$