CS1800
10/31 - Fri

Admin

- HW4 graded!
- HW5 due 11/14 9pm
- Tom - Elizabeth Peabody House w, Th, Sn - food pantry

Agenda

1. Sequences
2. Sequence Types
3. Deriving formulas

1. Sequences

↳ A sequence is a discrete structure used to discover, characterize patterns → Real-life: earthquakes, comets, stock market
used to study repeated processes

sequence notation/vocab
- ordered list of (possibly) infinite numbers

$\{a_n\}$ define a sequence → looks like a set, but:
- ordered
- dupes ok

notation

a_n - whole sequence / formula term

a_k - one term / arbitrary k

finite $\{a_n\} = a_1, a_2, a_3, \ldots, a_n$
↳ n terms

infinite $\{a_n\} = a_1, a_2, a_3, \ldots$

Infinite sequence: mapping from $\mathbb{Z}^+$ to $\{a_n\}$
$\mathbb{Z}^+$ values give us k, the term in the sequence

ex

$\mathbb{Z}^+$	1	2	3	4	5	...
$\{a_n\}$	2	4	8	16	32	...
	a_1	a_2	a_3	a_4	a_5	

a_k = double previous value
$2 \cdot a_{k-1}$
recursive. need prev terms

$\boxed{a_k = 2^k}$
closed form: just need k!
$a_{100} = 2^{100}$

Context: what we do with sequences
- given formula a_k, compute first few terms
  compute value of a_k for arbitrary k
- given the first few terms, derive a formula for a_k
  ↳ closed or recursive

ex $\{a_n\}\ \forall n \in \mathbb{Z}^+ \quad a_k = k/_{k+1}$

$a_1 = 1/2 \quad a_2 = 2/3 \quad a_3 = 3/4 \quad a_4 = 4/5 \quad a_5 = 5/6$

$a_{512} = 512/_{513}$

ex $\{a_n\}$ $\forall n \in \mathbb{Z}^+$ $a_k = (-1)^k$

$a_1 = -1$ $a_2 = 1$ $a_3 = -1$ $a_4 = 1$ $a_5 = -1$

$a_{325} = -1$

ex what is $a_4 = ?$

(1) $a_k = 2^{k-1}$

$a_4 = 2^3 = 8$

(closed)

(2) $a_k = \lfloor k^2/3 \rfloor$

$a_4 = \lfloor 16/3 \rfloor = 5$

(closed)

(3) $a_k = a_{k-1} + 6$

$a_1 = 5$

(recursive)

$a_4 = a_3 + 6$

$a_3 = a_2 + 6$

$a_2 = a_1 + 6$

↳ base case! $a_1 = 5$

$a_2 = 5 + 6 = 11$

$a_3 = 11 + 6 = 17$

$a_4 = 17 + 6 = 23$

never <u>need</u> a recursive formula

everything can be closed

- closed = easier to compute a_k
- recursive = easier to come up with the formula

2. Types of Sequences

↳ given first few terms of a sequence, come up with formula a_k

Three common sequence types where we know, understand the formulas

- Arithmetic
- Geometric
- Quadratic

Type 1 - Arithmetic → add same value every time

$\{a_n\} = a, a+d, a+d+d, a+d+d+d, \ldots$

$= a, a+d, a+2d, a+3d, \ldots$

a = initial value

d = common difference

ex Arithmetic, $a = 3$ $d = 12$

$\{a_n\} = 3, 3+12, 3+12+12, 3+12+12+12$

$3, 15, 27, 39, \ldots$

in general, given a, d, k

$$\boxed{a_k = a + d(k-1)}$$

recursive: $a_k = a_{k-1} + d$

Type 2 - Geometric → multiply the same value every time

$\{a_n\} = a, a \cdot r, a \cdot r \cdot r, a \cdot r \cdot r \cdot r, \ldots$

$= a, a \cdot r, a \cdot r^2, a \cdot r^3, \ldots$

a = initial value

r = common ratio

ex geometric $a = -2$, $r = 2$

$\{a_n\} = -2, -2 \cdot 2, -2 \cdot 2^2, -2 \cdot 2^3, \ldots$

in general, a, r, k

$$\boxed{a_k = a \cdot r^{k-1}}$$

$= -2, -4, -8, -16, \ldots$ recursive: $a_k = a_{k-1} \cdot r$

Type 3 - Quadratic → quadratic formula

$$a_k = ak^2 + bk + c$$

- need to solve for a, b, c

ex $a = 2,\ b = 3,\ c = 4$

$a_k = ak^2 + bk + c$
need to know a, b, c

$a_1 = a \cdot 1^2 + b \cdot 1 + c$
$= 2 + 3 + 4 = \boxed{9}$

$a_2 = 2 \cdot 2^2 + 3 \cdot 2 + 4$
$= 8 + 6 + 4 = \boxed{18}$

$a_3 = 2 \cdot 3^2 + 3 \cdot 3 + 4$
$= 18 + 9 + 4 = \boxed{31}$

3. Deriving a Formula

↳ given first few terms of a sequence, goal formula for a_k

Is it arith, geo, or quadratic?
↳ we know the formula!

↳ Is the sequence one of those 3 types?

- diffs between terms
  - same? arithmetic!
  - not same, but second-level diffs are same? Quadratic
- ratios between terms
  - same? geometric!

ex $\{a_n\} = 4, 1, -2, -5, -8, \ldots$ $a = 4$

diffs $-3\ \ -3\ \ -3\ \ -3$ same! Therefore, arithmetic $d = -3$

closed form $\boxed{a_k = 4 - 3(k-1) = 4 - 3k + 3 = 7 - 3k}$ → Plug in a, d to general arithmetic formula $a_k = a + d(k-1)$

recursive $a_k = a_{k-1} - 3$
$a_1 = 4$ (base case)

ex $\{a_n\} = 6, 2, 2/3, 2/9, 2/27, \ldots$

diffs $-4\ \ -1\tfrac{1}{3}\ \ -??$

not the same
not arithmetic !!

ratios $6/3=2$ $2/3=2/3$ $\frac{2/3}{3}=2/9$ divide by 3 every time geometric!

$a=6$ $r=1/3$

Closed form $a_k = 6\cdot\left(\frac{1}{3}\right)^{k-1}$ → Plug in a, r to general geometric formula $a_k = a\cdot r^{k-1}$

recursive $a_k = a_{k-1}\cdot\frac{1}{3}$

$a_1 = 6$

ex $\{a_n\} = 1, 3, 6, 10, 15, 21, \ldots$

diffs 2 3 4 5 6 ~ not arithmetic :(

2nd level 1 1 1 1 ~ 2nd levels are same therefore, quadratic!

we know $a_k = ak^2+bk+c$

- need to solve for a, b, c
- 3 variables need 3 equations

$a_1 = a\cdot 1^2 + b\cdot 1 + c$

$= a + b + c = 1$

$a_2 = a\cdot 2^2 + b\cdot 2 + c$

$= 4a + 2b + c = 3$

$a_3 = a\cdot 3^2 + b\cdot 3 + c$

$= 9a + 3b + c = 6$

solve the system of equations:

$$\begin{array}{r} 4a + 2b + c = 3 \\ -(a + b + c = 1) \\ \hline 3a + b = 2 \end{array}$$

b in terms of a:

$b = 2-3a$

$$\begin{array}{r} 9a + 3b + c = 6 \\ -(4a + 2b + c = 3) \\ \hline 5a + b = 3 \end{array}$$

↳ in terms of a

$5a + (2-3a) = 3$

$2a = 1$

$\boxed{a = 1/2}$!!

now we know a, can compute b

$b = 2-3a$

$= 2-3\cdot 1/2$

$= 2-3/2$

$\boxed{b = 1/2}$!!

now we know a, b solve for c

$a+b+c=1$

$1/2+1/2+c=1$

$\boxed{c=0}$!!

Put it all together:

$a_k = \frac{1}{2}k^2 + \frac{1}{2}k$

$= \frac{k^2+k}{2} = \boxed{\frac{(k)(k+1)}{2}}$

now we can compute a_k for any k

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$1, 3, 6, 10, 15, 21, \underline{28}$

↳ pattern

$$a_k = \frac{(k)(k+1)}{2}$$

$$a_7 = \frac{7 \cdot 8}{2} = \frac{56}{2} = \boxed{28}$$

formula

$a_1 = 1$

$a_2 = 1 + 2 = 3$

$a_3 = 3 + 3 = 6$

$a_4 = 6 + 4 = 10$

$a_5 = 10 + 5 = 15$

recursive $a_k = a_{k-1} + k$

$a_1 = 1$

triangle sequence

(famous!!)