CS1800
10/28-Tues.

Admin
- HW5 out, due 11/4 9pm
- Quizzes back EOD

Agenda
1. conditional probability
2. Bayes Theorem
3. Prob/Counting Practice

0. Review - Expected Value

Throw 4 balls into 3 bins, uniformly at random

1 2 3

What is the expected number of empty bins?

X_i = bin i

1 bin i is empty

0 bin i is not empty

linearity of expectation

↳ total # of empty bins X

$$E[X] = \underbrace{E[X_1]}_{\text{bin 1}} + \underbrace{E[X_2]}_{\text{bin 2}} + \underbrace{E[X_3]}_{\text{bin 3}}$$

$$= 3 \cdot E[X_i]$$

one bin, one throw. Pick bin 1

↳ $2/3 \cdot 1 + 1/3 \cdot 0 = 2/3$

one bin, 4 throws.

$(2/3)^4 = 16/81$

↳ all bins, all throws

$3 \cdot (16/81) = \boxed{.593}$

1. Conditional probability

→ reminder $Pr(E \cap F) = Pr(E) \cdot Pr(F)$ iff E, F are ind.

↳ today: E, F are not independent
- directly compute $|E \cap F|$
- apply Bayes rule

$$\boxed{Pr(E \mid F) = \frac{Pr(E \cap F)}{Pr(F)}}$$

F ↳ given

- we care about E
- F gives us more info

ex) bit strings of length 4 are equally likely $|S| = 16$

E = at least 2 consecutive zeroes

Pr(E)?

• 4 cons. zeroes	0000	(1)
• 3 cons. zeroes	1000 0001	(2)
• 2 cons. zeroes	1001 001_ _100	(5)

total $|E| = 1 + 2 + 5 = 8$

$\boxed{Pr(E) = 8/16 = 1/2}$

ex) E = at least 2 cons. zeroes F = starts with zero Pr(E|F)?

$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$ $Pr(E \cap F)$? $Pr(F)$?

$|E \cap F| = 5$ $Pr(E \cap F) = 5/16$

$|F| = 8$

$Pr(F) = 8/16$

$\underbrace{}_{S}$
0000
0001
0010
0011
0100

$Pr(E|F) = \frac{5/16}{8/16} = \frac{5}{16} \cdot \frac{16}{8} = \boxed{\frac{5}{8}}$ went up!

Summary $Pr(E|F)$ = prob of E, given F
F gives us more info

cannot! $Pr(E \cap F) = Pr(E) \cdot Pr(F)$
not this time!

another thought...

↳ $Pr(E|F)$ change the sample space!

original S = all bit strings of length 4

new S = all bit strings that start with ∅ : 8 of these

5 have 2 cons. zeroes

$5/8$

2. Bayes Theorem

→ real world conditional probability

$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$ But what if... 1. we can't compute $Pr(E \cap F)$?
(and E, F not independent)

2. we don't know $Pr(F)$?

1. insight - solve for $Pr(E \cap F)$

$Pr(E \cap F) = Pr(E|F) \cdot Pr(F)$

$Pr(F \cap E) = Pr(F|E) \cdot Pr(E)$

$Pr(E \cap F)$
$Pr(F \cap E)$
same!

↳ 2. insight - event F, event E

- F happens with E, or F without E

$Pr(F) = Pr(F \cap E) + Pr(F \cap \neg E)$

$= Pr(F|E) \cdot Pr(E) + Pr(F|\neg E) \cdot Pr(\neg E)$

Ex WNBA LV Aces — star Aja Wilson

- Aja plays 30 of 44 games
- Aces win 75% of games Aja plays in
- Aces win 40% of games Aja doesn't play in

What do we know?

✓ • $Pr(A) = 30/44$
• $Pr(W) = ?$
• $Pr(A|W) = ?$
• $Pr(A|\neg W) = ?$
✓ • $Pr(W|A) = .75$
✓ • $Pr(W|\neg A) = .40$

A = Aja plays   W = Aces win

↳ 1. Probability of a win?

2. Probability Aja played, given a win?

$Pr(W) =$ win with Aja (or) win w/o Aja

$= Pr(W|A) \cdot Pr(A) + Pr(W|\neg A) \cdot Pr(\neg A)$

$= (.75)(30/44) + (.40)(14/44)$

$= .6387$

$Pr(A|W) = \frac{Pr(A \cap W)}{Pr(W)}$

$Pr(A \cap W) = \frac{Pr(W|A) \cdot Pr(A)}{Pr(W)}$

$= \frac{(.75)(30/44)}{.6387} = .8008$

3. Probability Wrapped

You are dealt 4 cards, and order doesn't matter. What is the probability that your hand contains **exactly** 2 threes (of any suit)?

sample space S $C(52,4)$

event space E → threes $C(4,2)$

→ non threes $C(48,2)$

$\frac{C(4,2) \cdot C(48,2)}{C(52,4)}$

$\approx 2.5\%$

4 flavors of donut, Laney picks 2

Assuming all outcomes are equally likely what is the probability that Laney gets exactly two Chocolate donuts?

sample space S # stars = 2, # bars = 3 $C(5,2) = \frac{5!}{2!3!} = 10$

event space E only one outcome $|E| = 1$

$Pr(E) = 1/10$

.

Assuming all outcomes are equally likely, what is the probability that Laney gets **exactly two** Chocolate donuts, given that she got at least one chocolate donut?

E = exactly 2 choc

F = at least one choc

$Pr(E|F) = \frac{Pr(E \cap F)}{Pr(F)}$

$Pr(E \cap F) = Pr(E) = 1/10$

$Pr(F)$ – at least one choc

- exactly one → C1, C2, C3
- exactly two = 1 way

$Pr(F) = 4/10$

$Pr(E|F) = \frac{1/10}{4/10} = \frac{1}{4}$

Your Outlook filter tries to flag all of your spam emails. 5% of the emails you get are spam. Outlook flags 99% of the actual spam as spam. It flags 2% of the non-spam as spam. What is the probablity that an email is actually spam, given that Outlook flagged it?

S = actual spam

F = outlook flagged

$Pr(S) = .05$

$Pr(F)$?

$Pr(S|F) = ?$

$Pr(S|\neg F) = ?$

$Pr(F|S) = .99$

$Pr(F|\neg S) = .02$

What is question asking for?

$\hookrightarrow P(S|F)$

Conditional prob $Pr(S|F) = \dfrac{Pr(S \cap F)}{Pr(F)}$ need to break down both

$$Pr(S \cap F) = Pr(F|S) \cdot Pr(S)$$
$$= (.99)(.05)$$

$$Pr(F) = Pr(F|S) \cdot Pr(S) + Pr(F|\neg S) \cdot Pr(\neg S)$$
$$= (.99)(.05) + (.02)(.95)$$

plug back into cond formula

$$\frac{(.99)(.05)}{(.99)(.05) + (.02)(.95)}$$

$$= .7226$$