CS1800
10/21-Tue.

Admin

- HW4 due 9pm
- HW3 graded by EOD
- Quiz 2 Fri in lecture
- Quiz Review: 4-5pm HS108
  5-7pm HS102
- 1802 survey: tinyurl.com/1802survey

Agenda

1. Independent Events
2. Expected Value
3. Linearity of Expectation

0. Review

- sample space
  all possible outcomes of an experiment S
- event space
  $E \subseteq S$
- Pr(E) $\frac{|E|}{|S|}$

Draw a card
Pr(club or spade)
Pr(club) + Pr(spade)
$\frac{13}{52} + \frac{13}{52} = \frac{1}{2}$

roll two dice
Pr(4 and 5) (4,5)
Pr(4) · Pr(5)
$\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

1. Independent Events

↳ generalization of product rule, sum rule

events E, F ... generally $Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

iff E, F are independent $Pr(E \cap F) = Pr(E) \cdot Pr(F)$

what are independent events?

$Pr(E \cap F) = Pr(E) \cdot Pr(F)$

ex Pr(4) · Pr(5) independent - roll two dice
↳ roll two dice, the outcome of one does not impact the outcome of the other

ex Draw two cards, no replacement dependent

① Are E, F independent?
② Given that E, F are independent, compute $Pr(E \cap F)$

ex Flip a coin 3 times

order matters TTH ≠ THT

- we know: indv flips are independent
- fair coin: one flip 1/2 tails
  1/2 heads

S = {TTT, TTH, THT, HTT,
THH, HTH, HHT, HHH}

E = first flip is tails
F = odd # tails
Are E, F independent?

E = {TTT, TTH, THT, THH}
F = {TTT, THH, HTH, HHT}

$Pr(E \cap F) = Pr(TTT, THH) = 2/8 = 1/4$ ☑

$Pr(E) \cdot Pr(F) = \frac{4}{8} \cdot \frac{4}{8} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ ☑ yes!

↳ given independent events, compute various probabilities

ex Drawer A: 10 pink, 5 orange socks
Drawer B: 16 pink, 8 orange socks

$Pr(E \cup F) = Pr(E) + Pr(F) - Pr(E \cap F)$

$Pr(E \cap F) = Pr(E) \cdot Pr(F)$

Picking two socks?
- both same drawer...
- one from A, one B ... independent!

From this experiment:
- Prob of A-pink, B-pink?
  A-pink, B-orange?
- Prob of pink-A overall?
- Prob of orange-B overall?

Pr(pink)?
Pr(orange)?

joint probability table

	Drawer A- Pink	Drawer A - Orange	Pr(Drawer B)
Drawer B - Pink	$2/3 \cdot 2/3 = 4/9$	$1/3 \cdot 2/3 = 2/9$	$4/9 + 2/9 = 6/9$
Drawer B - Orange	$2/3 \cdot 1/3 = 2/9$	$1/3 \cdot 1/3 = 1/9$	$2/9 + 1/9 = 3/9$
Pr(Drawer A)	$4/9 + 2/9 = 6/9$	$2/9 + 1/9 = 3/9$	

A = pink, B = pink
A = pink, B = orange

marginal probabilities
- Pr (color from drawer)?

Pr (pink from A)?

ex pink - pink
pink - orange

Pr(any pinks?) = $4/9 + 2/9 + 2/9 = 8/9$

Pr (all orange?) = $1/9$

2. Expected Value

run an experiment over and over and over...
- on average, what happens?
- how often do certain outcomes happen?

(CS1800 version)

to answer what happens on average:
- Random Variable (not random, not a variable)
- numeric value assoc. w/ the outcome of an experiment

needs to be a number so we can compute an average

can't be: pink sock
Queen of H

- random variable: X

ex rolling a die
$Pr(X = 6) = 1/6$

- expected value $E[X]$

formula: s_i outcome
X_i value of the outcome

$$E[X] = \sum_{s_i \in S} Pr(s_i) \cdot X_i$$

sigma notation: adding terms!

$$E[X] = \sum_{s_i \in S} Pr(s_i) \cdot X_i = Pr(s_1) \cdot X_1 + Pr(s_2) \cdot X_2 + Pr(s_3) \cdot X_3 + \ldots + Pr(s_n) \cdot X_n$$

ex rolling a die $S = \{1, 2, 3, 4, 5, 6\}$ X = assoc. w/ the value on the die

$$E[X] = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = 3.5$$

↳ Pr(roll one) · value of a one

ex experiment: flip a coin three times
outcomes: TTT, THT, HHT, ...

what is $E[X]$ if X's...

X = associated with number of heads.

$Pr(X=3) = 1/8$
$Pr(X=2) = 3/8$
$Pr(X=1) = 3/8$
$Pr(X=0) = 1/8$

$E[X] = \frac{1}{8}\cdot 3 + \frac{3}{8}\cdot 2 + \frac{3}{8}\cdot 1 + \frac{1}{8}\cdot 0$

$= 3/8 + 6/8 + 3/8$

$= 1.5$

X = associated with number of distinct outcomes.
TTT = 1 THT = 2

$Pr(X=2) = 6/8$

$Pr(X=1) = 2/8$

$E[X] = \frac{6}{8}\cdot 2 + \frac{2}{8}\cdot 1$

$= \frac{12}{8} + \frac{2}{8}$

$= 1.75$

3. Linearity of Expectation

↳ what if we wanted to know the #heads if we flip a coin 100 times?

$Pr(X=0)$
$Pr(X=1)$
$Pr(X=2)$
...
$Pr(X=100)$

$\vdots$ n

so much work!

X_i for $i = 1, 2, 3, \ldots, n$

$E(X_1 + X_2 + X_3 + \ldots + X_n) = E[X_1] + E[X_2] + E[X_3] + \ldots + E[X_n]$

ex roll a die 100 times
X assoc. with sum of all rolls, on average

E(roll 1 + roll 2 + roll 3 + ... + roll 100)

E[roll 1] + E[roll 2] + E[roll 3] + ... + E[roll 100]
3.5 + 3.5 + 3.5 + ... + 3.5

$E(\text{all } 100) = 100 \cdot 3.5 = 350$

ex flip a coin 3 times : Indicator RV (1 = yes it happens) (0 = no it doesn't)
X assoc. with # heads

↳ break down into one flip $E[X_1] = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 0 = \frac{1}{2}$

↳ 3 flips $3 \cdot E[X_i] = 3 \cdot \frac{1}{2} = 1.5$ heads tails

$\therefore$ ◡

↳ 100 flips $100 \cdot E[X_i] = 100 \cdot \frac{1}{2} = 50$