CS1800
11/18 - Tues.

Admin

- HW7 due 12/2 9pm
- Second chance HW
  due 12/5 9pm
  (no 48 hour extension)
- Quiz 3 out today
- 11/25 optional lecture
  ↳ poll on pizzza

Agenda

1. Induction Review
2. Inequality ex.
3. Structural ex.

Lecture question

Will we need to be able to do a proof on the quiz?

→ yes! Quiz 4 12/5
one question
30-minute quiz (+ buffer)

1. Induction Review

For induction proofs, how is it considered "proven" if it is based on the ASSUMPTION that the p(k) is true? What about if it wasn't true?

predicate P(x)
↳ no time, no interest P(1), P(2), P(3), P(4)... could prove all separately

induction is a shortcut
↳ P(k) ⇒ P(k+1)
If P(k) is true, then P(k+1)
IH, assumed
↳ prove, using IH

It needs to be proven true at least once!
↳ Base case!
ex: show P(1)

Steps of a proof: ① Predicate ② Logic statement ③ Base case ④ Inductive step
Clearly labelled: IH, what is P(k)?, goal what is P(k+1)?, applying IH

Categories where induction is useful:
1. Summation
2. divisibility
3. inequality
4. structural
5. program correctness

Summation

- def of summation
- formula

prove: formula always works!

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \dots + n \quad \text{(def)}$$

$= \frac{(n)(n+1)}{2}$ (formula)

divisibility

- expression is a multiple of some integer
- if x is a multiple of y, then: $\exists w$ such that $w \cdot y = x$
- If x is a multiple of y, then so is $2x$, $3x$, etc.
- If x, z are multiples of y, then xz is a multiple, $x+z$ is a multiple

structural

- discrete structure property
- set, sequence, graph, etc.
- induction on the size of the structure

inequality

- some expression < other expression
- ex: $f(n) < g(n)$
- n is input to a function
- induction on n $(1, 2, 3, \dots)$
- we can do anything as long as it's still true! ❤

$100x < 100y$

$100x < 200y$

$100x < 200y^2 + 185000$

revisit to divis. proof from last time

$6^x + 4$ is a multiple of 5

Predicate	$P(x)$ *states that* $6^x + 4$ *is a multiple of* 5
Logic statement	$\forall n \in Z^+ \; P(n)$

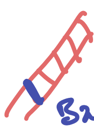
~ $\exists w \in \mathbb{Z}$ such that $5 \cdot w = 6^x + 4$

~ induction on x

Base case

$P(1)$ $6^1 + 4$ is a multiple of 5

$6^1 + 4 = 10$ True!



Base case

expression: $6^x + 4$

plug in 1: $6^1 + 4 = 10$

10 is a multiple of 5 $10 = 5 \cdot 2$

Inductive step $P(k) \Rightarrow P(k+1)$ Implication

Inductive Hypothesis $P(k)$, assumed true: $(6^k + 4)$ is a multiple of 5.

Goal: show $P(k+1)$, i.e., $(6^{k+1} + 4)$ is a multiple of 5.

$6^{k+1} + 4$ is multiple of 5 ~ need to prove!

Start with the expression first:

$(6^{k+1} + 4)$	$=$	$6^k \cdot 6^1 + 4$	// exponent law $a^b a^c = a^{b+c}$
	$=$	$6^k \cdot (5 + 1) + 4$	// 6 = 5+1. We do this because 5 is useful!
	$=$	$6^k \cdot 5 + \underline{6^k \cdot 1 + 4}$	// Distribute the 6^k

$6 = 5 + 1$ b/c useful!

$6^k \cdot 1 + 4$is a multiple of 5, by our Inductive Hypothesis.

$6^k \cdot 5$ is multiple by 5, by definition of multiplication

The sum of two multiples of 5 is itself a multiple of 5. Done!

IH: $6^k + 4$ is multiple of 5 ☺

$6^k \cdot 5$ is multiple of 5

add!

$\hookrightarrow 6^k(5+1)+4$

$= \underbrace{6^k \cdot 5}_{\text{mult. of } 5} + \underbrace{6^k \cdot 1 + 4}_{\text{mult. of } 5 \text{ (IH)}}$

2. Inequality Proof

- Predicate $P(x)$ $\quad 5x+5 \le x^2$
- Logic Statement — when is this true?

  $\forall n \in \mathbb{Z}^+ \; n \ge 6 \Rightarrow P(n)$

P(1) False!
P(2) False!
P(3) False!
P(4) False!
P(5) False!
P(6) True :)
↳ and up!

- Base case: prove P(6)

  LHS: $5 \cdot 6 + 5 = 35$ $\quad 35 \le 36$ ☑
  RHS: $6^2 = 36$

- Inductive Step: $P(k) \Rightarrow P(k+1)$

  IH $P(k)$, ie, $5k+5 \le k^2$ — assume
  goal $P(k+1)$ $\quad 5(k+1)+5 \le (k+1)^2$ — want to prove

start with LHS:

$$
\begin{aligned}
5(k+1)+5 &= 5k+5+5 && \text{// multiply 5 through} \\
&\le k^2 + 5 && \text{// by IH} \\
&\le k^2 + k && \text{// b/c } k \text{ is at least } 6 \\
&\le k^2 + 2k + 1 && \text{// b/c } k < 2k+1 \\
&= (k+1)^2
\end{aligned}
$$

done!

3. Structural Proof

↳ property of a discrete structure (set, sequence, graph)

previous set proofs:
- set equality, two sets w/ same elements

induction:
- set property, ignore specific elements

ex sets!
- induction on cardinality of the set
- P(1) — set with one element
  P(2) — set with two elements
  ...

Focus on: subsets of a set, (of size 2)

How many subsets of size 2?

$P(x)$ $|A| = x$, A has $\frac{(x)(x-1)}{2}$ subsets of size 2.

$A = \{a, b\}$
Subsets of size 2: 1

$A = \{a, b, c\}$
Subsets of size 2: 3

$A = \{a, b, c, d\}$
Subsets of size 2: 6

Logic Statement: $\forall n \in \mathbb{Z}\ n \geq 2 \Rightarrow P(n)$ ~ could also use $\forall n \in \mathbb{Z}^+$

Base case: $|A| = 2$ $P(2)$
one 2-element subset, A itself

Formula: $\frac{(2)(2-1)}{2} = 1$ ☑

Inductive Step: $P(k) \Rightarrow P(k+1)$

IH $P(k)$ is true: $|A| = k$, then A has $\frac{(k)(k-1)}{2}$ 2-element subsets

Goal $P(k+1)$ $|A| = k+1$, then A has $\frac{(k+1)(k)}{2}$ 2-element subsets

- Let A be a set with $k+1$ elements, $|A| = k+1$
- remove one element, x, from A, to get set A'

$A' = A - \{x\}$

$A = \{-, -, -, \ldots, -, x\}$ $|A| = k+1$

$A' = \{-, -, -, \ldots, -\}$ $|A'| = k$

A' has $\frac{(k)(k-1)}{2}$ 2-element subsets by IH

- element x is not in any of A' subsets
- create 2-element subsets $\{x, \_\}$ ~ How many are there?
↓
pair up x with every element in A'
↓
there are k of them!

↳ total 2-element subsets of A is 2-element subsets of A' + $\{x, \_\}$

total 2-element subsets in A:

A' 2-element: $\frac{(k)(k-1)}{2}$ $\{x, \_\}$: k

$$\frac{(k)(k-1)}{2} + k$$ // take subsets of size 2

$$= \frac{(k)(k-1)}{2} + \frac{2k}{2}$$ // common term

$$= \frac{k^2 - k + 2k}{2}$$ // terms together

$$= \frac{k^2 + k}{2}$$

$$= \frac{(k)(k+1)}{2}$$ done!

Goal was:

2-element subsets in A

$$\frac{(k)(k+1)}{2}$$