Professor Hamlin
Day 17

Agenda

1) Admin

HW 7 released - shorter and only 80 pts

Exam 2 next Friday in class

- practice problems released → try doing them before looking at the answer.

updated the practice problems just now, 1PM on Nov 8, to include them).

- **Bayes Rule will not be given on exam2, you should memorize this (worth your while down the road)**
- Binomial / Poisson distribution formulas won't be needed
- an image of the equations is included below

$E[x] = \sum_x x * P(X = x)$

$\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

	No Repeat Selections	Repeat Selections
Order Matters	Permutations: $P(n,k) = \frac{n!}{(n-k)!}$	Product Rule: n^k
Order Doesn't Matter	Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$	Stars and Bars (Partitions of k identical objects into n groups): $\binom{k+n-1}{n-1}$

2) Exam Content

3) Summation notation

4) Strong induction

5) Induction: inequality

Exam contents

→ Counting style probability e.g $\frac{|E|}{|S|}$

→ Expected Value / Variance

→ Bayes rule problem

→ BFS / DFS

→ Dijkstra algorithm

→ induction (equality or inequality)

Review: induction

Show: $x^0 + x^1 + \ldots + x^n = \frac{1-x^{n+1}}{1-x}$

Base case: $n = 0$

$x^0 = \frac{1-x^1}{1-x}$

$1 = 1$ ✓

Conditional

$P \to Q$

1. Assume P

$P \to \cdot \to \cdot \to Q$

WLOG - without loss of generality

Inductive Step: if statement n, then statement n+1
P Q

P: $x^0 + x^1 + \dots + x^n = \frac{1-x^{n+1}}{1-x}$

Q: $x^0 + x^1 + \dots + x^n + x^{n+1} = \frac{1-x^{(n+1)+1}}{1-x}$

Proof: Assume P is true (Inductive Hypothesis)
start w/

$$x^0 + x^1 + \dots + x^n + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} \quad \text{(by I.H)}$$

$$\frac{1-x^{n+1}}{1-x} + \frac{x^{n+1}(1-x)}{1-x}$$

$$\frac{1-x^{n+1} + x^{n+1} - x^{n+1}\cdot x}{1-x}$$

$$\frac{1-x\cdot x^{n+1}}{1-x}$$

$$\frac{1-x^{(n+1)+1}}{1-x}$$

Summation Notation

We've been writing $1+\dots+n$ → want quicker way: Summation!

$$1+2+4+8+16+32+64$$

$$= 2^0+2^1+2^2+2^3+2^4+2^5+2^6$$

Note have 7 terms

$$= \sum_{k=0}^{6} 2^k$$

6 ← last value of k

starting value of k

→ by putting each value of k into this term, then sum them

"The sum of 2^k where k goes from 0 to 6"

Exercise Express the following in sum notation

1) $9+11+13+15+17$

$\sum_{k=0}^{4} 2k+9$ or $\sum_{k=1}^{5} 2k+7$

$\sum_{k=4}^{8} 2k+1$

2) $1+2+3+4+5+\ldots+n$

$\sum_{k=1}^{n} k$ or $\sum_{k=0}^{n-1} k+1$

Compute each sum below:

1) $\sum_{k=10}^{12} 2k$

$2\cdot10+2\cdot11+2\cdot12$

66

2) $\sum_{k=0}^{101} (-1)^k$

pattern ↓

0 → 101

102 terms

$\underbrace{1+-1}_{0 \quad 1^{st}} + \ldots + \underbrace{1+-1}_{100 \quad 101}$

0

Using summation in induction

from last class: Show $1+2+\dots+n = \frac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

then $1+2+\dots+n+n+1 = \frac{(n+1)(n+2)}{2}$

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

starting with ... $\sum_{k=1}^{n} k + (n+1)$ equivalent

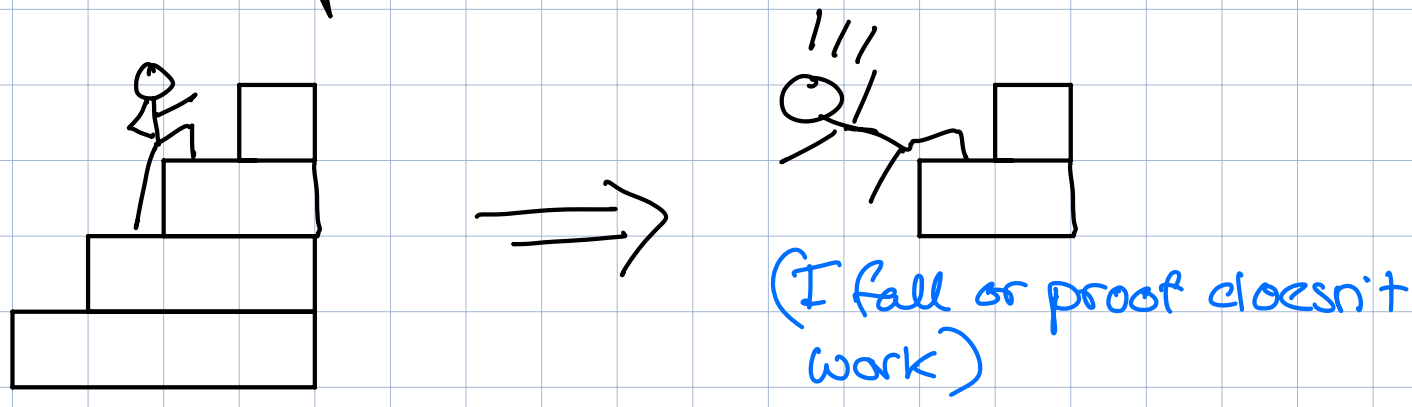
Can always expand summation notation

Back to induction

So far we've only needed to assume the previous step held

on a ladder, only need to assume previous rung

on stairs what happens if I don't assume previous steps exist?



(I fall or proof doesn't work)

weak: if statement n, then statement n+1

strong: if statement 1, 2, ... n then statement n+1

Example: Using only 3¢ and 4¢ coins can make any change starting at 6¢

11¢ = 2 4¢ + 1 3¢ 19¢ = 4·4¢ + 1·3¢

Base Cases:

n=6	3¢ + 3¢
n=7	3¢ + 4¢
n=8	4¢ + 4¢

previously assumed just n!

Induction Step: if statement 6, 7, ... to n are true statement n+1 is true

Assume some combo of 3¢ & 4¢ produce n=6 ... n

Show for n+1:

OOOO(3¢) → OOOO(4¢)

Case 1: n's sum includes a 3¢ coin → switch out to 4¢ to make n+1

n-4

Case 2: n's sum only 4¢ coins →

n-8

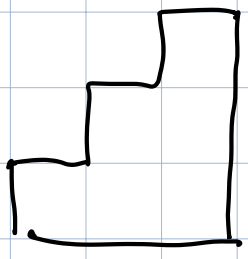
must be at least 2, 4¢ (by base case)
replace 2 4¢ w/ 3 3¢ for n+1

Strong Induction

Like weak induction, prove S(1), S(2) ....
but we use stronger assumption

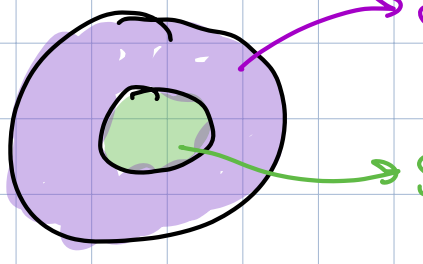
Weak → just assume previous rung exists

Strong → assume every step before exists



Process: 1) Prove statement for 1st few n
2) Show that S(1), S(2), ... S(n) implies S(n+1)

When should you use weak vs. strong?



→ set of problems can prove w/ strong induction

→ set of problems can prove w/ weak induction

Can always use strong, though if can use weak

might be a simpler proof.

Exercise Suppose two algorithms compute same task but take a different # of steps to do so

Algorithm 1: on input of size $n \rightarrow 2^N$ steps
Algorithm 2: on input of size $n \rightarrow N!$ steps

input size	1	2	3	4	5	6
Algo 1	2	4	8	16	32	64
Algo 2	1	2	6	24	120	720

1) Complete the table
2) Which algo for $n=2$ - 2
3) Which algo for $n=5$ - 1
4) Which algo for any size n? 1

It grows slower! Can we prove it?

(weak) **Induction w/ inequalities**

"Prove that $2^N < N!$ for all N above 4"

Recall

Induction Recipe (from previous lesson)
1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if S(n) then S(n+1)
4. Prove inductive step:
   a. assume statement n (inductive hypothesis)
   b. write statement n + 1 in two halves
      (tip: start at sum side, work to other side)
   c. apply assumption to get from one half to other

Statement: $2^N < N!$

Base case: $N = 4$

$2^4 < 4! \Rightarrow 16 < 24$ ✓

Inductive step if statement n (P), then n+1 (Q).

P: $2^N < N!$

Q: $2^{N+1} < (N+1)!$

$2^3 = 2 \cdot 2 \cdot 2$
$2 \cdot 2^2$

Proof: Assume P (inductive hypothesis)

starting with 2^{N+1}

$= 2 \cdot 2^N$

$< 2 \cdot N!$ (by I.H.)

$< (N+1) N!$ (by $2 < N+1$ due to base case)

$< (N+1)!$

$2 < N+1$

So this was some handwaving right? Sort of there are certain things we can do w/ inequalities we can't do w/ equalities

Move 1 add same thing to both sides, doesn't change inequality

if $3 < 4$ then $3+10 < 4+10$

$x < y \rightarrow x + c < y + c \quad \forall \; c \in \mathbb{R}$

Move 2 multiply by positive value, doesn't change inequality

if $3<4$ then $3 \cdot 10 < 4 \cdot 10$

if $x<y$ then $x \cdot c < y \cdot c \quad \forall\ c \in \mathbb{R} \wedge c > 0$

Move 3: multiply by negative value, swaps inequality direction

if $3<4$ then $3 \cdot -1 > 4 \cdot -1$

if $x<y$ then $xc > yc \quad \forall x, y, c \in \mathbb{R}$ w/ $c<0$

Move 4: sum two inequalities (large side together & small side together)

if $3<4$ AND $5<6$ then $3+5 < 4+6$

if $x<y$ AND $w<z$ then $x+w < y+z$

Move 5: can replace smaller side of inequality w/ something smaller (or larger side w/ larger)

$z < N+1$

if $3 < 7$ and $1 < 3$ then $1 < 7$

if $y < z$ and $x < y$ then $x < y$

$2 \cdot N! < (N+1)N!$

Hint: one of most common manipulations for induction

Exercise Complete table

	1	2	3	4	5	6
N^2	1	4	9	16	25	36
$N+10$	11	12	13	14	15	16

Show $N+10 < N^2$ for all $N \geq 4$ above some value

Proof Base case $N=4$

$4+10 < 16$ ✓

Inductive step if statement n (P) then statement n+1 (Q)

P: $N+10 < N^2$

Q: $(N+1)+10 < (N+1)^2$

Proof Assume P (I.H.)

$(N+1)+10 = (N+10)+1$

$< N^2 + 1$

Know $N^2 + 2N + 1 > N^2 + 1$

$< N^2 + 2N + 1$

$< (N+1)^2$

for $N > 4$

ONLY WORK LHS OF Q

clear