Professor Hamlin
Day 18

Agenda

1) admin → Exam 2 this friday (this material not on exam)
→ no quiz this week
2) Review
3) Series & Sequences
   - Arithmetic
   - Geometric
   - Quadratic

1) Recognize
2) i-th term
3) Partial sum

Review Weak induction on inequalities

1: $x<y$, $x+c<y+c$

2: $x<y$, $x\cdot c<y\cdot c$ if $c>0$

3: $x<y$, $x\cdot c>y\cdot c$ if $c<0$

4: $x<y$ $w<z$ $x+w<y+z$

5: $x<y$ $z<x$, $z<y$

Strong induction · stars vs ladder
assume $S(1), S(2), \ldots S(n)$ vs assume $S(n)$

Exercise Which are valid manipulations of inequalities ~~(and what move?)~~

1) $x+10<z \overset{?}{\Rightarrow} x<z$
valid

2) $x<y,\ y<z \overset{?}{\Rightarrow} x<z$
valid

3) $x-10<y \overset{?}{\Rightarrow} 10-x<-y$
invalid $10-x>-y$

Last thing from last class: summation

← last value of k

$$\sum_{k=0}^{4} 1+2^k = \begin{array}{l} (1+2^0) \\ +(1+2^1) \\ +(1+2^2) \\ +(1+2^3) \\ +(1+2^4) \end{array} = \begin{array}{l} 1 \\ +3 \\ +5 \\ +9 \\ +17 \end{array} = 35$$

↑ 1st value of k

k is the value always increasing by 1

Things to sum can come in some common formats

Think about:

1, 3, 5, 7, 9, ... vs 2, 6, 10, 14, ...

-increasing by + some #

1, 2, 4, 8, 16, ... vs 1, 3, 9, 27, ...

*multiplying by some constant

Feel similar but something like ...

2, 4, 6, 8, 10 ... vs 1, 4, 9, 25, 36

Grows at a very different rate! We can capture this idea formally. But first some vocab.

Sequence: an ordered list of objects

e.g. 1, 2, 3, 4, 5, 6 ....

Series: sum of an infinite sequence

e.g $1+2+3+4\ldots = \sum_{k=1}^{\infty} k$

Term: individual object in series / sequence

e.g 1, 2, 3, 4... 2 is 2nd term

Partial Sum: sum of just part of a series

e.g $1+2+3+4 = \sum_{k=1}^{4} k = 10$

First type of Sequence: Arithmetic

10, 12, 14, 16, 18, 20, ...

first difference +2 +2 +2 +2 +2

18, 12, 6, 0, −6, −12, ...

−6 −6 −6 −6 −6

What do these two sequences have in common?

the difference between terms is constant

(next term − current term)

How to tell if sequence is Arithmetic?

Check if difference between terms is constant

Arithmetic Series / Partial Sum

$$\underset{\underset{10+2\cdot 0}{\uparrow}}{\overset{k=0}{10}} + \underset{\underset{10+2\cdot 1}{\uparrow}}{\overset{k=1}{12}} + \underset{\underset{10+2\cdot 2}{\uparrow}}{\overset{k=2}{14}} + 16 + 18 + 20 + \ldots = \sum_{k=0}^{\infty} 10+2k$$

General form is:

$$\sum_{k=0}^{\infty} \overset{\text{1st term in sequence}}{a_0} + \underset{\text{difference between terms}}{d}k$$

Example 1 $\overset{k=0}{5} + \overset{k=1}{2} + \overset{k=2}{-1} + \overset{k=3}{-4} + \ldots$

$a_0 = 5$

$d = 2 - 5 = -3$

$$\sum_{k=0}^{\infty} 5-3k$$

Second type of sequence: Geometric

$$\frac{1}{2}, \xrightarrow{\times 2} 1, \xrightarrow{\times 2} 2, \xrightarrow{\times 2} 4, \xrightarrow{\times 2} 8, \xrightarrow{\times 2} 16, \ldots$$

$$100, \xrightarrow{\times -1/10} -10, \xrightarrow{\times -1/10} 1, \xrightarrow{\times -1/10} -1/10, \xrightarrow{\times -1/10} 1/100, \ldots$$

What do these two sequences have in common?

The ratio (next term/current term) is constant

How to tell sequence is Geometric? Divide next term by previous term and result is constant

Example: 100, -10, 1, ...

ratio: $\frac{-10}{100} = \frac{-1}{10}$ ratio: $\frac{1}{-10} = \frac{-1}{10}$

Geometric Series / Partial Sum

$$\overset{K=0}{\underset{\overset{\uparrow}{\frac{1}{2}\cdot 2^0}}{\frac{1}{2}}} \overset{\times 2}{\frown} + \overset{K=1}{\underset{\overset{\uparrow}{\frac{1}{2}\cdot 2^1}}{1}} \overset{\times 2}{\frown} + \overset{K=2}{\underset{\overset{\uparrow}{\frac{1}{2}\cdot 2^2}}{2}} + \overset{K=3}{4} + 8 + \ldots = \sum_{K=0}^{\infty} \frac{1}{2}\cdot 2^K$$

General Form is:

$$\sum_{K=0}^{\infty} \underset{\overset{\uparrow}{\text{starting term}}}{a_0}\, \underset{\overset{\uparrow}{\text{ratio of } \frac{\text{next term}}{\text{previous term}}}}{r}{}^{K}$$

Example/ $18 + 6 + 2 + \frac{2}{3} + \frac{2}{9}, \ldots$

$a_0 = 18$

$r = \frac{6}{18} = \frac{1}{3}$

$$\sum_{K=0}^{\infty} 18 \cdot \left(\frac{1}{3}\right)^K$$

Third type of sequence is: Quadratic

Harder to see than arithmetic/geometric

1, 3, 7, 13, 21, 31

Arithmetic? $1 \xrightarrow{+2} 3 \xrightarrow{+4} 7 \xrightarrow{+6} 13 \xrightarrow{+8} 21 \xrightarrow{+10} 31$ Nope!

Geometric? $1 \xrightarrow{1/3} 3 \xrightarrow{3/7} 7 \xrightarrow{7/13} 13 \xrightarrow{13/21} 21 \xrightarrow{21/31} 31$ Nope!

What they actually look like is ....

$$a_n = an^2 + bn + c$$

n-th term

constant values

Example; $a = 1$, $b = 0$, $c = 0$ e.g. $a_n = 1n^2 + 0 \cdot n + 0$

n=0	n=1	n=2			
0,	1,	4,	9,	16,	25,
$1 \cdot 0^2 + 0 \cdot 0$	$1 \cdot 1^2 + 0 \cdot 1 + 0$	$1 \cdot 2^2 + 0 \cdot 2 + 0$			

How do we identify Quadratic sequence/series?

1, 3, 7, 13, 21, 31

first diff. $\xrightarrow{+2}\ \xrightarrow{+4}\ \xrightarrow{+6}\ \xrightarrow{+8}\ \xrightarrow{+10}$

Second diff: $\xrightarrow{+2}\ \xrightarrow{+2}\ \xrightarrow{+2}\ \xrightarrow{+2}$

The second difference is constant

Exercise 1 Identify arithmetic, geometric, quadratic or none. If arithmetic or geometric write in sum notation

1) 6, 15, 28, 45, 66, 91

Quadratic

9 13 17 21 25

4 4 4 4

2) 1, −4, 16, −64, 256, ...

Geometric

x−4 x−4 x−4 x−4

$$\sum_{k=0}^{\infty} 1\cdot(-4)^k$$

k=0

$1\cdot(-4)^0$

$1\cdot 1$

3) 4, 7, 10, 13, 16, 19, ...

Arithmetic

+3 +3 +3 +3 +3

$$\sum_{k=0}^{\infty} 4+3k$$

4) 2, 7, 11, 42, −4

None

+5 +4 +31 + −46

−1 27

Coming back to getting a, b, c for Quadratic from the sequence

$$\underset{k=0}{6} + \underset{k=1}{15} + \underset{k=2}{28} + 66 + 91 + \ldots.$$

$$(a \cdot k^2 + bk + c)$$

$$\text{system of equations} \begin{cases} 6 = a \cdot 0^2 + b \cdot 0 + c & \Rightarrow \boxed{c = 6} \\ 15 = a \cdot 1^2 + b \cdot 1 + c \\ 28 = a \cdot 2^2 + b \cdot 2 + c \end{cases}$$

Solving system of equations:

$$6 = c$$
$$15 = a + b + c$$
$$28 = 4a + 2b + c$$

Substitute in c and simplify

$$15 = a + b + 6 \Rightarrow 9 = a + b$$
$$28 = 4a + 2b + 6 \Rightarrow 22 = 4a + 2b$$

Solve for a / b to substitute into other equation:

$$9 = a + b \Rightarrow b = 9 - a$$

Substitute into other eqn and simplify

$$22 = 4a + 2(9 - a)$$

$$22 = 4a + 18 - 2a$$
$$4 = 2a$$
$$a = 2$$

Substitute into original eqn:

$$b = 9 - a \Rightarrow b = 9 - 2 \Rightarrow b = 7$$

Thus, $a = 2$, $b = 7$, $c = 6$

Checking work: $a_n = 2n^2 + 7n + 6$ $n = 0$

$$6 + 15 + 28 + 66 + 91 + \ldots$$

$6 = 2 \cdot 0^2 + 7 \cdot 0 + 6$ ✓

$15 = 2 \cdot 1^2 + 7 \cdot 1 + 6$ ✓

$28 = 2 \cdot 2^2 + 7 \cdot 2 + 6$ ✓

Note we want to start Quadratic series at $k=0$ as it makes solving this easier

Exercise Find a, b, c for

$$1 + 3 + 7 + 13 + 21 + 31 + \ldots = \sum_{k=0}^{\infty} ak^2 + bk + c$$

$a \cdot 0^2 + b \cdot 0 + c = 1 \Rightarrow \boxed{c = 1}$

$a \cdot 1^2 + b \cdot 1 + c = 3$

$a \cdot 2^2 + b \cdot 2 + c = 7$

$$a+b=2 \longrightarrow b=2-a$$

$$4a+2b=6$$

$$4a+2(2-a)=6$$

$$2a=2$$

$$\boxed{a=1}$$

$$b=2-1=1$$

$$\boxed{\sum_{k=0}^{\infty} k^2+k+1}$$

Up next: Partial Sums (Arithmetic/Geometric)

Arithmetic:

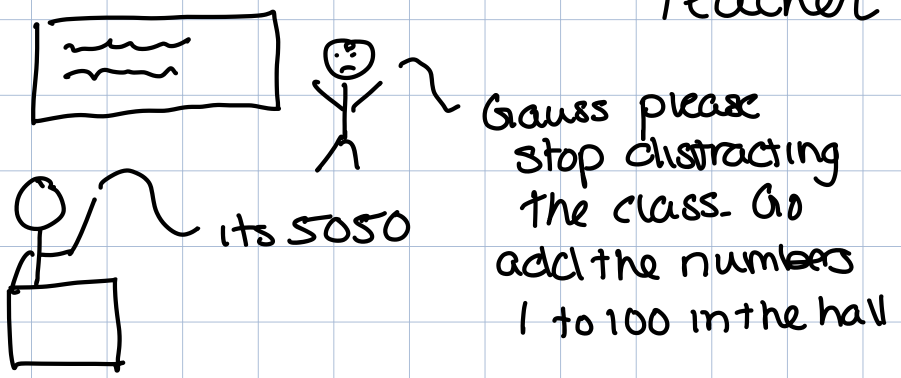
$$0+1+2+3+4=\sum_{k=0}^{4} k = ? \qquad 10$$

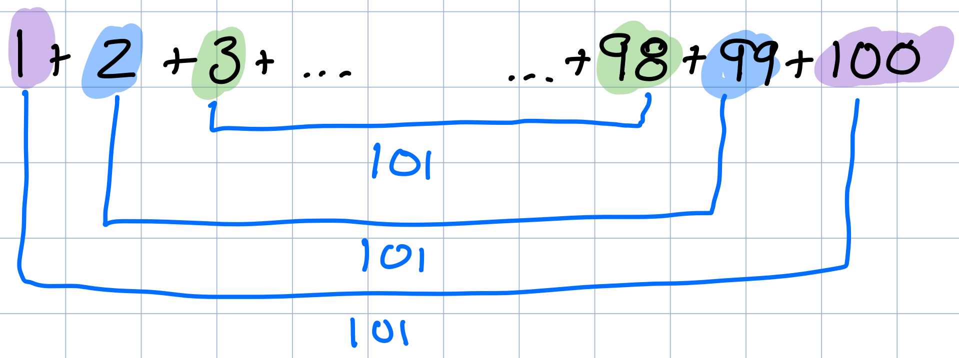
Geometric:

$$1+2+4+8+16=\sum_{k=0}^{4} 2^k = ? \qquad 31$$

Faster way than just summing everything up?
Yes!

Apocryphal tale: Gauss and a frustrated teacher





50 sums of 101 ⇒ $50 \cdot 101 = 5050$

General Form:

first term

last term

$$\sum_{k=0}^{N} a_0 + dk = (a_0 + a_N) \cdot \left(\frac{N+1}{2}\right)$$

number of pairs (rounded up)

Example: $\overset{N=0}{1} + \overset{1}{2} + \overset{2}{3} + \overset{3}{4} + \overset{4}{5} = 15$

$$\boxed{\sum_{k=0}^{4} 1 + k}$$

$a_0 = 1$

$a_N = 5$

$N = 4$

careful, 5 terms but we start at 0 so N=4

$$(1+5)\left(\frac{4+1}{2}\right) = \boxed{15}$$

Geometric Series Partial Sum

This can be a bit unintuitive for how we get this equation. Humor me for a moment

Partial Sum ↗ we want

$$S = \sum_{K=0}^{N} ar^{K} = a + ar + ar^2 + \dots + ar^{N-1} + ar^{N}$$

So let's compute $r \cdot S$, for fun....

$$r \cdot S = ar + ar^2 + ar^3 + \dots + ar^{N} + ar^{N+1}$$

So consider the following:

$$S - rS = (a + ar + ar^2 + \dots + ar^{N-1} + ar^{N})$$
$$- \, (ar + ar^2 + \dots + ar^{N-1} + ar^{N} + ar^{N+1})$$

all of these terms cancel out

Leaving $S - rS = a - ar^{N+1}$

Remember S is what we want to compute so we solve for S

$$\frac{S(1-r) = a - ar^{N+1}}{1-r}$$

Thus

$$\boxed{S = \frac{a_0(1-r^{N+1})}{1-r}}$$

Example $1+2+4+8+16 = 31$

$$\sum_{k=0}^{4} 1 \cdot 2^k \qquad S = \frac{1(1-2^{4+1})}{1-2} = \frac{-31}{-1} = \boxed{31}$$

remember even though 5 terms, largest value of k is 4

Summary of Arithmetic, Geometric & Quadratic

(k=0)

	Arithmetic	Geometric	Quadratic
How to identify	2 4 6 8 10 ... +2 +2 +2 +2 Difference constant	1 2 4 8 16 ... ×2 ×2 ×2 ×2 Constant ratio	1 3 7 13 21 2 4 6 8 +2 +2 +2 constant second difference
Expression of Single term	$a_0 + dk$	$a_0 \cdot r^k$	$ak^2 + bk + c$
Computing Partial Sum	$\sum_{k=0}^{N} a_0 + dk =$ $(a_0 + a_N)\left(\frac{N+1}{2}\right)$	$\sum_{k=0}^{N} a_0 r^k =$ $\frac{a_0(1-r^{N+1})}{1-r}$	Calculus fun (not CS1800)

Exercise

1) $\sum_{K=0}^{100} 4-K$

$a_0 = 4-0=4$

$a_{100} = 4-100 = -96$

$(4+-96)\left(\frac{100+1}{2}\right) = \boxed{-4646}$

2) $\sum_{K=0}^{10} 10\cdot 3^K$

$a_0 = 10\cdot 3^0 = 10$

$\frac{10(1-3^{11})}{1-3} = \boxed{885730}$

3) $10+7+4+1+(-2)+(-5)+(-8)$

$\sum_{K=0}^{6} 10-3K$

$a_0=10$

$a_6 = -8$

$N=6$

$(10+-8)\left(\frac{6+1}{2}\right) = \boxed{7}$

Summary of Arithmetic, Geometric & Quadratic

$(k=1)$

	Arithmetic	Geometric	Quadratic
How to identify	2 4 6 8 10 … +2 +2 +2 +2 Difference constant	1 2 4 8 16 … ×2 ×2 ×2 ×2 Constant ratio	1 3 7 13 21 2 4 6 8 +2 +2 +2 constant second difference
Expression of Single term	$a_0 + d(k-1)$	$a_0 \cdot r^{k-1}$	$ak^2 + bk + c$
Computing Partial Sum	$\sum_{k=1}^{N} a_0 + dk =$ $(a_0 + a_N)\left(\frac{N}{2}\right)$	$\sum_{k=1}^{N} a_0 r^k =$ $\frac{a_0(1-r^N)}{1-r}$	Calculus fun (not CS1800)