Day 10:

Admin:
- plan for Friday: practice exam for 1st half, student-motivated questions in the 2nd half
- exam instructions available (see piazza post)
- hw4 note:
    - no need to compute a final value, leave unsimplified (as HW instructions indicate)
- hw4 dates:
    - due Friday @ 11:59 PM
    - late due date is Saturday @ 11:59 PM
    - solutions are available Sunday @ 12:10 AM

Content:
- combinations
- leftover principle
- counting partitions of identical objects

Ordering: when does it matter?

Order matters:

How many ways can a student take 3 CS courses from 10 unique courses?

(CS 1800, DS 2000, DS 2500)

$\neq$

(DS 2500, DS 2000, CS 1800)

TUPLE

Order doesn't matter:

How many ways can one take 3 candies from 10 unique candies?

{CHOC, LOLLY, GUMMY}

=

{LOLLY, GUMMY, CHOC}

SET

Over-counting (multiplicative)

How many people are in the room if …

… there are 100 eyes in the room
… there are 90 fingers in the room
… there are 400 limbs (legs & arms) in the room

Punchline:
If there are n items (eyes, fingers, limbs)
and c items per every item-of-interest (people)
then there are n / c items of interest

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? $C = \{1, 2, 3\}$
(order doesn't matter)

3 WAYS:

$\{1,2\}$ $\{1,3\}$ $\{2,3\}$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? $C = \{1, 2, 3\}$
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING TWO ORDERED CANDIES:

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? C = {1, 2, 3}
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING TWO ORDERED CANDIES:

THERE ARE $2! = 2$ WAYS OF ORDERING 2 CANDIES

(1, 2) (1, 3) (2, 3)
(2, 1) (3, 1) (3, 2)

OVERCOUNTING (MULTIPLICATION):

WAYS OF CHOOSING 2 FROM 3 (ORDER NOT MATTER) = WAYS OF ORDERING 2 FROM 3 (ORDER MATTERS) / WAYS OF ORDERING 2 (ORDER MATTERS)

1 MIN

HOW MANY WAYS CAN YOU SELECT

5 CARDS FROM 52

ORDER MATTERS

52 51 50 49 48

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{47!} = P(52,5)$$

ORDER DOESN'T MATTER

$$\frac{P(52,5)}{5!}$$

AC 2C 3C 4C 5C

2C AC 3C 4C 5C

5!

Combination: definition & formula

- A combination is a subset of objects (order doesn't matter)
  (how many ways can I choose k items from n possible)

- A permutation is an ordering of objects (order matters)
  (how many ways can I order k items from n possible)

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n choose k"

$\binom{n}{k}$ AKA Binomial coefficient

In Class Activity

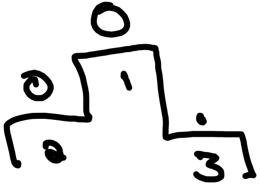
ORDER MATTER?

REPEAT AN ITEM?

How many ways can the 8 Mario Kart racers form the final podium of 3 winners. The order of the podium matters.

$M, L, P \neq L, M, P$

1 2 3



$8 \cdot 7 \cdot 6 = P(8,3)$

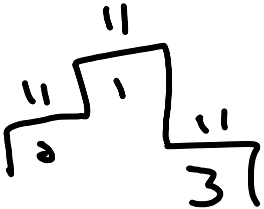
ORDER MATTER: YES

REPEATS: NO

How many ways can the teams (mercedes, ferrari, etc) arrange on the podium of 3 winners in a formula 1 race? (assume that each team has at least 3 cars in the race and there are 11 teams total)

M F

example outcome: (1st place: Mercedes, 2nd place: Mercedes, 3rd place: Ferrari)



MMF

$11 \cdot 11 \cdot 11 = 11^3$

ORDER MATTER: YES

REPEATS: YES

How many 5 card hands exist in a deck of 52 unique cards? ("hands" are unordered)

ORDER MATTER: NO

REPEAT: NO

$$\frac{P(52,5)}{5!} = \binom{52}{5}$$

How many 47 card hands exist in a deck of 52 unique cards?

$$\binom{52}{47}$$

AC, 2C, 3C, 4C, 5C

2C, AC, 3C, 4C, 5C

Compute a final number for the two problems above, how (and why?) are they related?

$$\binom{52}{5} = \frac{52!}{(52-5)!\,5!} = \frac{52!}{47!\,5!}$$

$$\binom{52}{47} = \frac{52!}{(52-47)!\,47!} = \frac{52!}{5!\,47!}$$

Combinations: Leftover principle

How many ways can I choose 10 student to take out for ice cream, from this class of size n?

How many ways can I choose n - 10 students to leave out of my ice cream party?

Combinations: Leftover principle

WAYS OF CHOOSING K OBJECTS FROM n

$$\binom{n}{k} = \binom{n}{n-k}$$

WAYS OF CHOOSING n-K OBJECTS TO "LEAVE BEHIND"

For every selection of k items, there is another selection of n-k items which is not chosen.

Counting: Putting it together (almost ... see later slide for complete version of this table)

HOW TO SELECT K ITEMS FROM N

ABC
CAB

FAMILY PORTRAITS
PODIUM

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$	MYSTERY (FOR NOW)

PASSWORD

DRAW 5 CARDS IN POKER

How many ways can 2 friends share 8 slices of pizza?

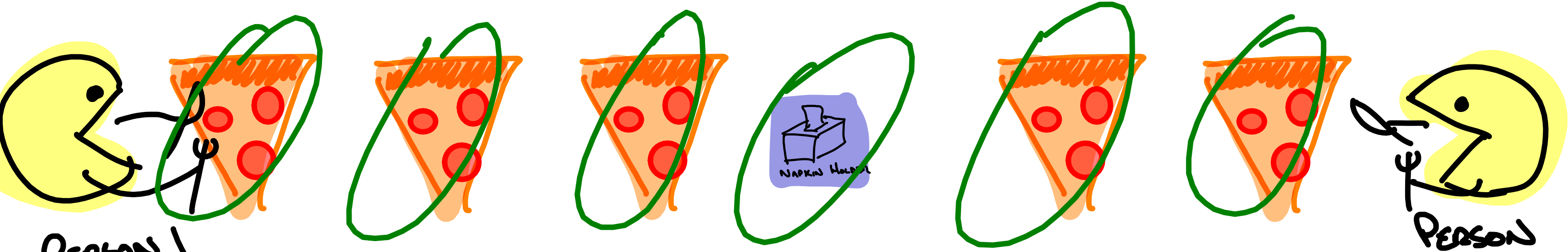
(0,8), (1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1), (8,0)

FRIEND A GETS 0 SLICES OF PIZZA

FRIEND B GETS 8

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can two people split k slices of pizza?



PERSON 1
EATS ALL PIZZA BEFORE BOUNDARY

PERSON 2
EATS ALL PIZZA AFTER BOUNDARY

K SLICES PIZZA

1 BOUNDARY

THERE ARE $\binom{K+1}{1}$ WAYS

$$\binom{n+k-1}{k-1}$$

$n = 8$ SLICES PIZZA
$k = 2$ FRIENDS

$$= \binom{8+2-1}{2-1} = \binom{9}{1} = \frac{9!}{(9-1)!\,1!} = \frac{9!}{8!}$$

$$= 9$$

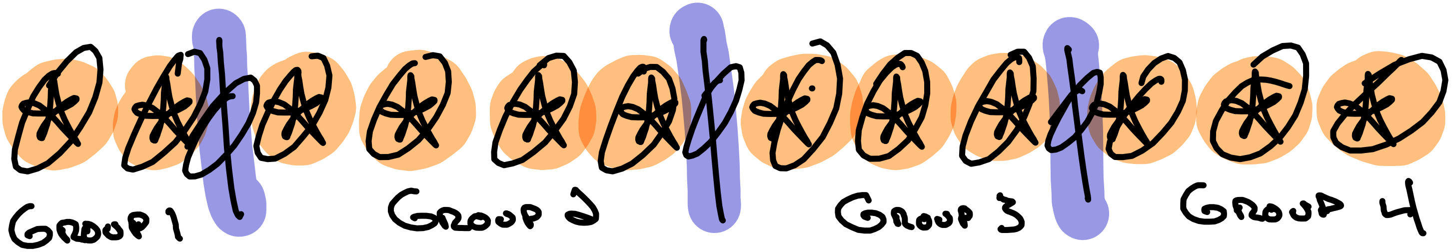
Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~two~~ THREE people split K slices of pizza?

$\binom{K+2}{2}$ WAYS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~the people~~ N GROUPS split k ~~s[illegible]~~ STARS?



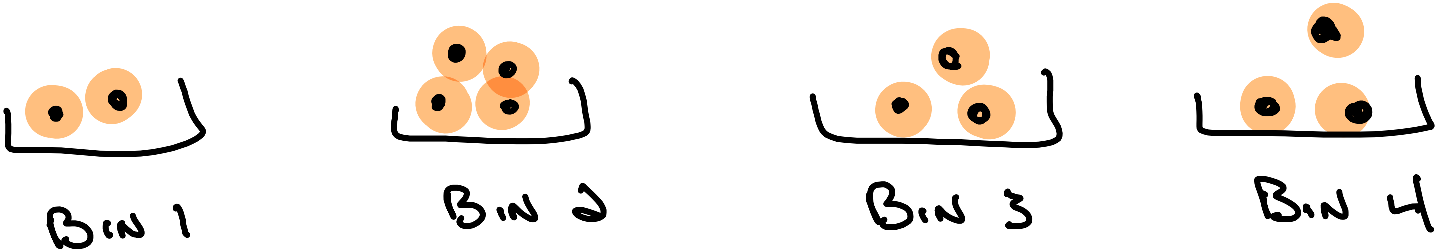
$\binom{K+N-1}{N-1}$ WAYS

NEED N-1 BOUNDARIES FOR N GROUPS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can t[redacted] split k s[redacted]?

N BINS BALLS



$$\binom{K+N-1}{N-1}$$ WAYS

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$ How many tuples of length k can one make from N items? (no repeats)	PRODUCT RULE N^K How many tuples of length k can one make from N items? (repeats)
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$ How many sets with k unique items can one make from N items? (no repeats)	PARTITION OF IDENTICAL ITEMS (STARS + BARS / BALLS IN BINS) $\binom{K+N-1}{N-1}$ How many ways can we split k identical items among N groups?

How is the balls-in-bins fit into bottom right box of "putting it together"?

Old problem: How many ways can we split 8 slices of pizza between two friends?

We're selecting from set of 2 friends 8 times:

aaaaaaaa (friend-a gets all the pizza)

aaaaaaab or aaaaaaba or aaaaabaa or ... (friend-b gets 1 slice, friend-a gets 7)

aaaabbbb or abababab or .... (friends-a gets 4 slices and friend-b gets 4 slices)

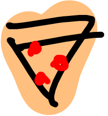
notice that:

- we may repeat selections
  (a, b may show up more than once) ✓

- the order of our selections doesn't matter
  (aaaabbbb is the same as abababab)

Two conventions for stars and bars

In class now:

K items 

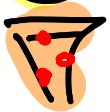
N groups 

$$\binom{K+N-1}{N-1}$$

Appears this way in summary chart

More common:

K groups 

N items 

$$\binom{N+K-1}{K-1}$$

While we're making counting review materials:

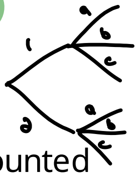
Counting Fundamentals:

- Principle of Inclusion-Exclusion (PIE): Counting the union of sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Product Rule: How many tuples can be made pulling first item from A and next from B?

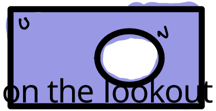
$$|A \times B| = |A| \times |B|$$



Counting moves:

- Count-by-partition: Partition items we want to count into subsets which are more easily counted (remember: each item to be counted shows up in exactly one subset)

- Count-by-complement: Count items not-of-interest, subtract it from "everything"



$$|U - N| = |U| - |N|$$

- Count-by-simplification: Be on the lookout for simpler, equivilent problems

M, L, P

P, L, M

Counting advice:

1. Clearly document your thinking on the paper
   (you'll clarify your thinking and find errors)

2. If you're stuck:
   - head back to the materials of the past few slides (PAGE 18+20)
   - try solving a simpler "sub-problem", the experience may provide fresh insight
     - (often useful for count-by-partition)

3. BUILD A FEW OUTCOMES

In Class Activity:

ORDER MATTERS?

REPEAT ITEMS?

How many passwords of length 5 can be made from vowels (upper and lowercase)?

How many ways can I select 10 students in this room to give a million extra credit points to? (assume 200 students in room)

5 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged? (assume 5 countries each have 1 swimmer each)
(e.g. in tokyo 2020 it was 1. Australia, 2. Hong Kong, 3. Canada)

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.

(++) redo the swimming problem, but assume that 5 countries each have 2 swimmers each.

In Class Activity:

How many passwords of length 5 can be made from vowels (upper and lowercase)?

ORDER MATTERS? YES

REPEAT ITEMS? YES

aaaoa

oaaaa

UaUaU

$N = 10$ VOWELS

$K = 5$ NUMBER OF SELECTION

_ _ _ _ _

10^5

In Class Activity:

How many ways can I select 10 students in this room to give a million extra credit points to? (assume 200 students in room)

ORDER MATTERS? No

REPEAT ITEMS? NO

$\{ S_7 \; S_8 \; S_4 \; S_{100} \; S_4 \ldots \}$

$N = 200$

$K = 10$

$$\binom{200}{10} = C(200, 10)$$

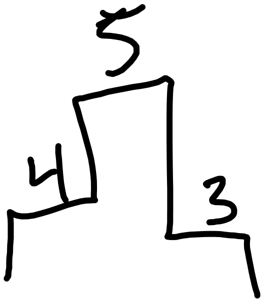
$$\frac{P(200,10)}{10!} = \frac{200!}{(200-10)!\,10!}$$

In Class Activity:

ORDER MATTERS? YES
REPEAT ITEMS? NO

5 countries each have one woman swimming in the women's 200m freestyle. How many ways ~~might the podium's nationality be arranged?~~ (assume 5 countries each have 1 swimmer each)
(e.g. in tokyo 2020 it was 1. Australia, 2. Hong Kong, 3. Canada)

1. HONG KONG 2. AUSTRALIA 3. CANADA



1. Aus 2. Aus 3. Aus

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = 5\cdot4\cdot3$$

In Class Activity:

ccccccc... √ ≠ √ cccccc...

ORDER MATTERS? NO

REPEAT ITEMS? YES

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

CCCCC

PPPPP

$\rightarrow$ CC, PP = PPCC

CV PC

3^{14}

$$\binom{N+K-1}{N-1} = \binom{16}{2}$$

K = 14

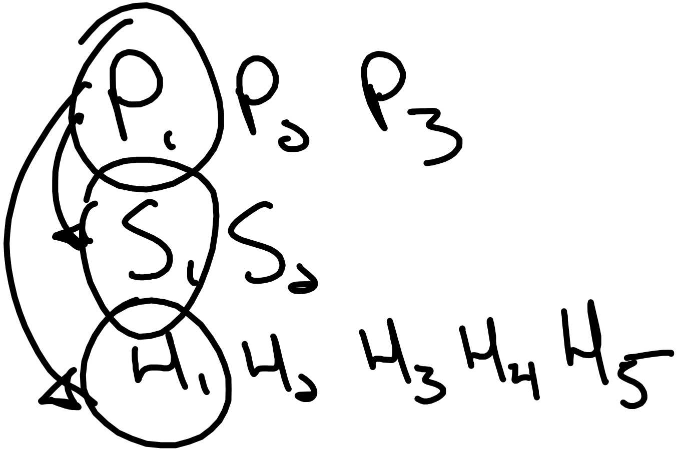
N = 3

$$\circ\circ\circ\circ \mid \circ\circ\circ\circ\circ \mid \circ\circ\circ\circ\circ$$

ORDER MATTERS?
REPEAT ITEMS?

In Class Activity:

I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?



$$|P \times S \times H| = |P| \times |S| \times |H|$$

VALID OUTFITS = TOTAL OUTFITS POSSIBLE − INVALID OUTFITS

$$= 3 \cdot 2 \cdot 5 - 2 - 5 + 1$$

$P_1 \cap H_1$ $\quad$ $P_1 \cap S_1$

$$\frac{P_1}{P} \quad \frac{S_1}{S} \quad \frac{H_1}{H}$$

SECRET DIE

$$\text{OUTFITS}_{\text{INVALID}} = \left\{ x \in P \times S \times H \;\middle|\; \begin{array}{c} x \notin P_1 \cap H_1 \\ \text{OR} \\ x \notin P_1 \cap S_1 \end{array} \right\}$$

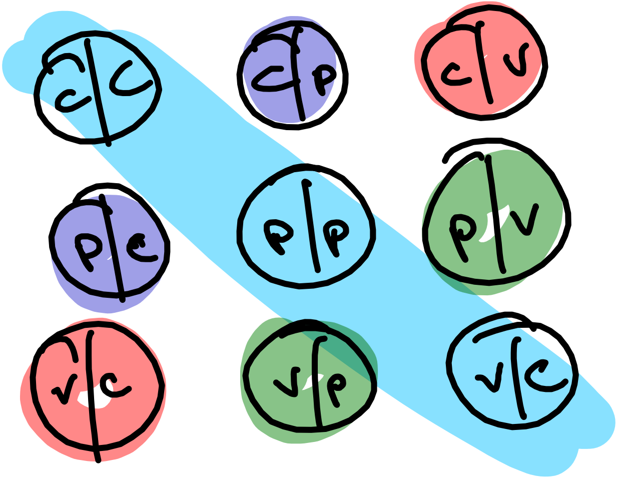
$$(P_1, S_2, H_5)$$

In Class Activity:

ORDER MATTERS?
REPEAT ITEMS?

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.



PIZZAS 1 TOPPING

$$\binom{3}{1} = 3$$

PIZZAS 2 TOPPINGS

$$\binom{3}{2} = 3$$

In Class Activity:

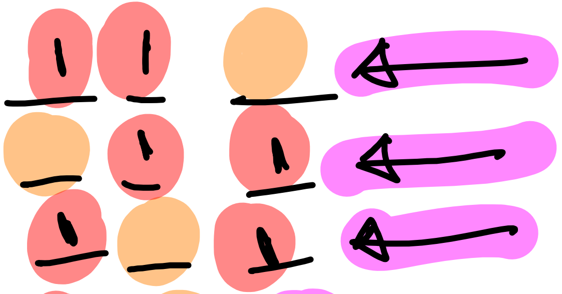
ORDER MATTERS?
REPEAT ITEMS?

5 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged? (assume 5 countries each have 1 swimmer each)
(e.g. in tokyo 2020 it was 1. Australia, 2. Hong Kong, 3. Canada)

(++) redo the swimming problem, but assume that 5 countries each have 2 swimmers each.

UNIQUE PODIUM + 1-REPEAT PODIUM

$P(5,3)$



$5 \cdot 4 \cdot 3$