Agenda
 1) Admin
       Practice exam
       Exam instructions
       Leave answers in factorial form (Hw)
 2) Review
 3) Combinations (order doesn't matter)
              - left over principle
 4) Balls & Bins

Review

	Can reuse options (e.g passwords)	Can't reuse options (e.g. people in photo)
Permutations (order matters)	n^k	$P(n,k) = \dfrac{n!}{(n-k)!}$

Counting stragettes - count by partition, count by complement, simplification

Exercise: 1) How many orders can I pet 5 cats at a cat cafe, if I can pet a cat more than once? ↳ with 10 cats total

$n=10$ $k=5$ $\boxed{10^5}$

2) Same question, but now I can't pet a cat more than once? $P(10,5)$ $\dfrac{10!}{5!}$

Combination - (order does not matter)

Grab a fist of two candies
from the halloween candy bowl
(Jolly Rancher, Reeseo Cup,
Miky way)

J   R
M

How many different combinations?

| JR | JM | R M |
| RJ | MJ | M R |

In ordered pairs this is $P(3,2) = \dfrac{3!}{1!} = 6$

But I'm grabbing a fistful at a time →
if doesn't matter what order I grab
them in !

this is
the same
when I don't
consider
order

(J, R)   (J, M)   (R, M)   3 ways
(R, J)   (M, J)   (M, R)   unordered

We've overcounted, we have 2x the count.
OR there are 2! = 2 ways of ordering
2 candies

Informally combinations are:

$$
\begin{array}{ll}
\text{Ways of} & = & \dfrac{\text{Ways of ordering 2 from 3 } (\cancel{\text{order matters}})}{\text{ways of ordering 2 } (\cancel{\text{order matters}})} \\
\text{choosing} & \\
\text{2 from 3} & \\
\text{(order not} & \\
\text{matter)} &
\end{array}
$$

Formally: choosing k from n items, order doesn't matter

$$
C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}
$$

↑
"n choose k"

	Can reuse options (e.g passwords)	Can't reuse options (e.g. people in photo)
~~Permutations~~ order matters	n^k	$\dfrac{n!}{(n-k)!}$
~~Combinations~~ order doesn't matter	? Come back to this	$\binom{n}{k} = \dfrac{n!}{(n-k)\,k!}$

Exercise 1) How many ways can 8 mariokart racers form the top 3. The order of 1st, 2nd, & 3rd matters

order matters, no repeats
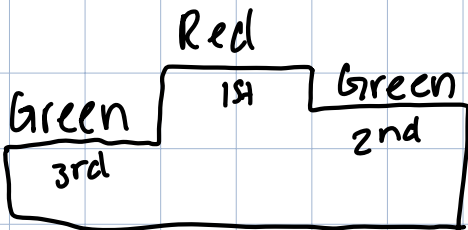
TOAD 1st
PEACH 3rd
YOSHI 2nd
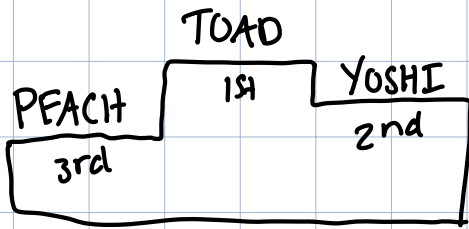
$$P(8,3) = \boxed{\dfrac{8!}{5!}}$$

2) We now have 4 teams in mariokart, with 3 members each. How many different team orderings can we have for 1st, 2nd, 3rd?

order matters, yes repeats

$$\underline{4}\ \underline{4}\ \underline{4} = 4^3$$

Red 1st
Green 3rd
Green 2nd

3) How many unique hands of 5 cards can be made from standard 52 card deck? (Hands are unordered)

order doesn't matter, no repeats

$n = 52$ $k = 5$

$$\binom{52}{5} = \dfrac{52!}{5!\,(52-5)!} = \boxed{\dfrac{52!}{5!\,47!}}$$

4) How many ways can one select the "remaining" 47 cards from the problem above?

$n = 52$ $k = 47$

$$\binom{52}{47} = \dfrac{52!}{47!\,(52-47)!} = \boxed{\dfrac{52!}{47!\,5!}}$$

Leftover principle

How many ways can I choose all but 10 students to take out for ice cream in class of size n?

$$\binom{n}{10} = \boxed{\frac{n!}{(n-10)!\,10!}}$$

How about the n-10 to take out for ice cream?

$$\binom{n}{n-10} = \frac{n!}{(n-(n-10))!\,(n-10)!} =$$

$$\frac{n!}{(\cancel{n}-\cancel{n}+10)!\,(n-10)!} = \frac{n!}{(n-10)!\,10!}$$

It doesn't matter who we choose to <u>include</u> or <u>exclude</u>, it is the same # of combinations

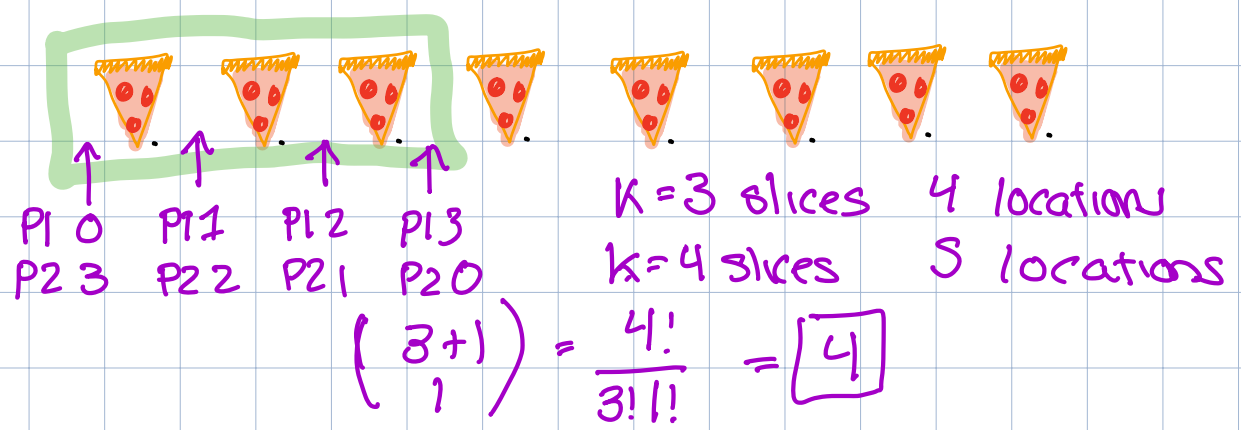|	Can reuse options (e.g. passwords)	Can't reuse options (e.g. people in photo)
Permutations order matters	n^k	$\frac{n!}{(n-k)!}$
Combinations order doesn't matter	? Come back to this	$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Counting partitions of identical objects

(Balls & Bins or Stars & Bars)

How many ways can two people split k slices of pizza?

will eat everything to left of box

pizza box

will eat everything to right of box
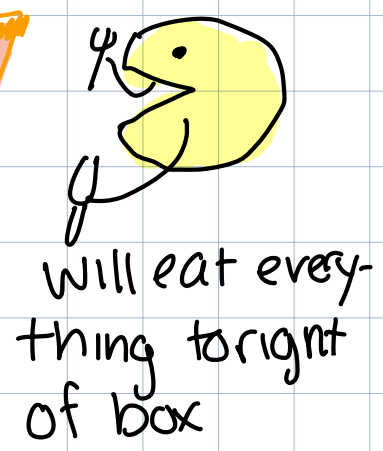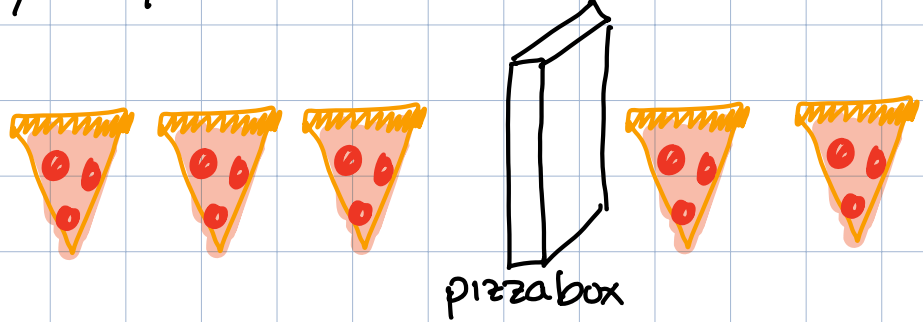
1. Pizza slices are identical so they can be in any order
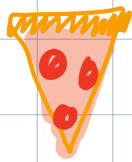2. What varies is the number of pizza each eats, i.e. where we put the barrier

P1 0   P1 1   P1 2   P1 3
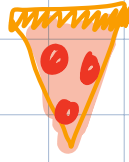P2 3   P2 2   P2 1   P2 0

K=3 slices   4 locations
k=4 slices   5 locations

$$\binom{3+1}{1} = \frac{4!}{3!\,1!} = \boxed{4}$$

k slices of pizza

1 boundary

$$\binom{k+1}{1} \text{ ways}$$

Three people?

will eat everything to left of box

pizzabox
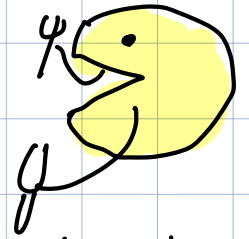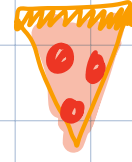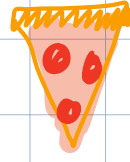
pizzabox

Will eat anything between boundary

Will eat everything to right of box

$$\binom{k+2}{2} \text{ ways}$$

# of slices → k

2 → # of boundaries

Formally if we have n piles and k items

order doesn't matter
can repeat

people → n piles (Bins) & (Bars)

pizza slices → k items (Balls) & (stars)

stars & bars

Group 1   Group 2   Group 3 ↑ Group 5

Group 4
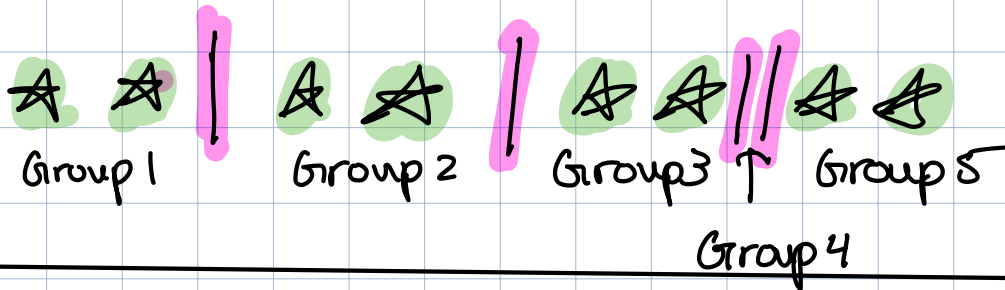
balls & bins
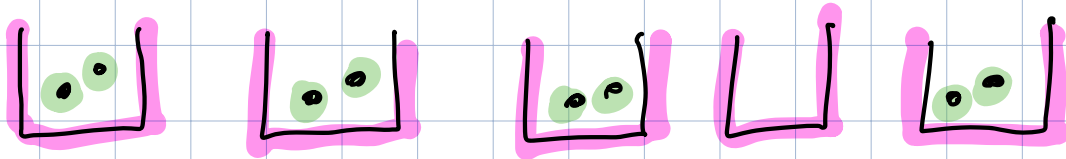
Need $n-1$ boundaries for n piles

$$\binom{k+n-1}{n-1} \text{ ways}$$

Putting it together

	Can reuse options (e.g passwords)	Can't reuse options (e.g. people in photo)
Permutations (order matters)	n^k	$P(n,k) = \dfrac{n!}{(n-k)!}$
Combinations (order doesn't matter)	$\dbinom{k+n-1}{n-1}$	$\dbinom{n}{k} = \dfrac{n!}{(n-k)k!}$

Example:

How many ways split candy? eg 9/1/0   10 pieces of candy

balls = k = 10 $\dbinom{10+3-1}{3-1}$ $\dbinom{12}{2}$

bins = n = 3

Warning two <u>equivalent</u> formulas for balls & bins

K Item 🍕 N groups	K Item 🍕 N groups
$\dbinom{K+N-1}{N-1}$	$\dbinom{N+K-1}{K}$
(what we used)	
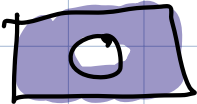
Equivalent due to leftover principle

Counting in summary:

- Sum rule: an ~or~ of options
  - if disjoint $(A \cup B) = |A| + |B|$
  - if not, use PIE : $|A \cup B| = |A| + |B) - |A \cap B|$

- Product rule: an ~and~ of options
  $$|A \times B| = |A| * |B|$$

- Counting strageties
  - Count-by-partition: break into disjoint subsets, combine using sum rule
  - Count-by-complement: count items not of interest $\quad |U - I| = |U| - |I|$
  - Count-by-simplification: look for simplier, equivalent problems

Advice:

1. Clearly document approach (easier to find mistakes)
2. If stuck, review counting rules / strageties
   try solving simplier sub problem
   determine if order matters & are repeats allowed

1) How many passwords of length 5 can be made from vowels (upper and lowercase)?

$$A E I O U \qquad \underline{10}$$

· order matters, can repeat

$$\underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10} \qquad \boxed{10^5}$$

2) How many ways can I select 10 students in this room to give a million extra credit points to each?

(125 students)   order doesn't matter
                 can't repeat

$$n = 125$$
$$k = 10$$

$$\binom{125}{10} = \boxed{\frac{125!}{10!\,115!}}$$

3) 10 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

order matters, no repeats $n = 10\ k = 3$

$$P(10,3) = \boxed{\frac{10!}{7!}}$$

4) How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

order doesn't matter
can repeat

$$Balls = k = 14$$
$$Bins = n = 3$$

$$\binom{14+3-1}{3-1} = \binom{16}{2} = \boxed{\frac{16!}{14!\,2!}}$$
$$= \binom{16}{14}$$

I terrorized a NEU Building for an hour

6) I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?

i.e. won't wear pant A w/ shirt B and pant A w/ hat C

complement → partition

$$\overline{P}\ \ \overline{S}\ \ \overline{H}$$

U = all possible outfits

I = shirt B w/ pant A ~or~ pant A w/ hat C

$$|U| = \frac{3}{P} \cdot \frac{2}{S} \cdot \frac{5}{H} = 30$$

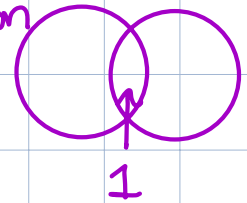$$|U| - |I| = \boxed{24}$$

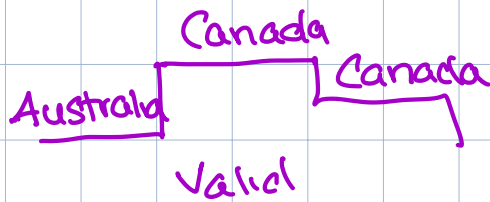$$|I| = \left|A_P \text{ w/ } B_S\right| + \left|A_P \text{ w/ } C_H\right| - 1 = 6$$

$$\frac{1}{P} \cdot \frac{1}{S} \cdot \frac{5}{H} \qquad \frac{1}{P} \cdot \frac{2}{S} \cdot \frac{1}{H}$$

$$\frac{A}{P}, \frac{B}{S}, \frac{C}{H}$$

↑ intersection

1

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7) ~~5~~ countries each have ~~one~~ woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada) (partition)

Canada
Canada
Australia
Valid

No nation repeated
(order matters, no repeat)

$$\frac{5 \cdot 4 \cdot 3}{}$$

$$P(5, 3)$$

nation is repeated
(trickier)

$$\frac{A}{}\ \frac{A}{}\ \frac{B}{} \qquad \frac{5 \cdot 4}{} \Big\}$$
$$\frac{A}{}\ \frac{B}{}\ \frac{A}{} \qquad \frac{5 \cdot 4}{} \Big\}$$
$$\frac{B}{}\ \frac{A}{}\ \frac{A}{} \qquad \frac{5 \cdot 4}{} \Big\}$$

$$5 \cdot 4 + 5 \cdot 4 + 5 \cdot 4$$

$$3 \cdot (5 \cdot 4)$$

8) **5** countries each have **one** woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

No nation repeated	2 repeated	3 times
" "	" "	$\underline{A}\ \underline{A}\ \underline{A}\ \underline{5}$

$$\boxed{\frac{5!}{3!} + 5\cdot3\cdot4 + 5}$$

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.

Bins are still pizza types, we just have more options

CP   CV   PV
CC   VV   PP

6 bins, 14 pizza $\dbinom{14+6-1}{6-1} = \dbinom{19}{5}$