1) Admin

- Hw 8 due today
- Hw 9 due next Tuesday
- Exam3 next Tuesday

2) Review

3) Merge Sort

4) Recurrences

Professor Hamlin
Day 21

Review Search

Linear $O(n)$
Binary $O(\log n)$

Sorting

Insertion $O(n^2)$

Runtime: # of comparisons in worst case

Exercise How many comparisons needed to sort the list below?

2 3 4 8 7 6

8 comparisons

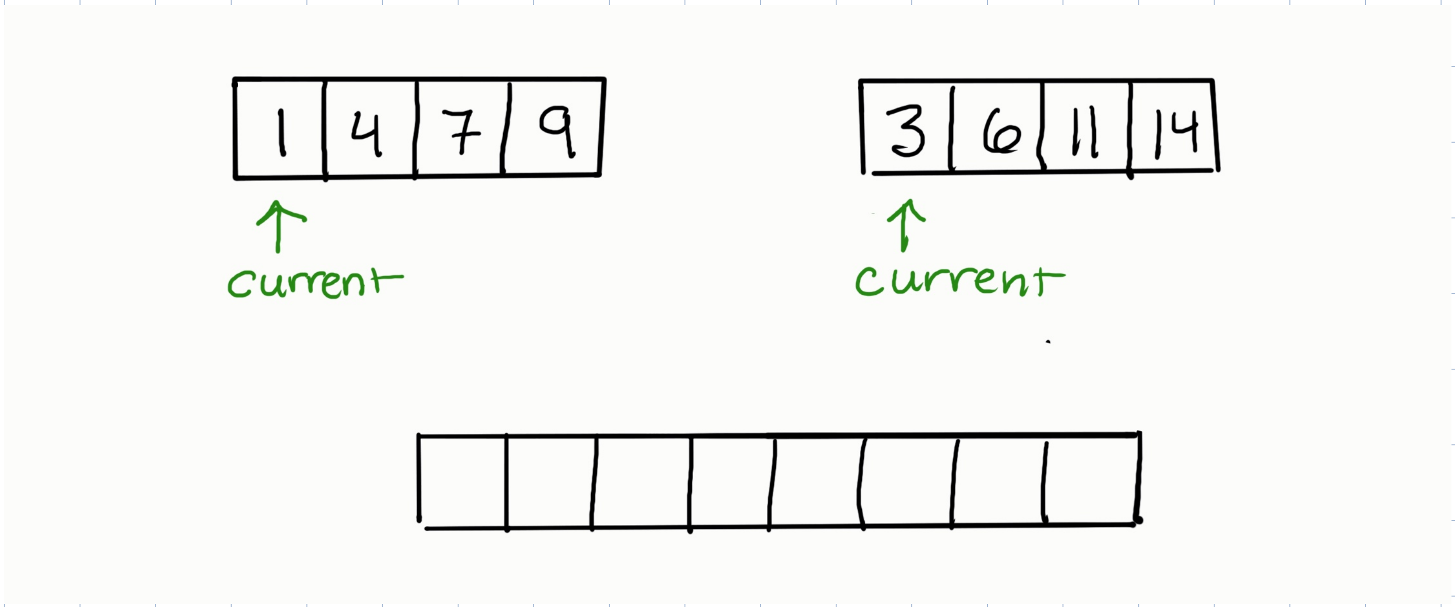
Recall Insertion sort requires $O(n^2)$ runtime

$$T(n) = n^2$$

Can we do better?

Merge Sort

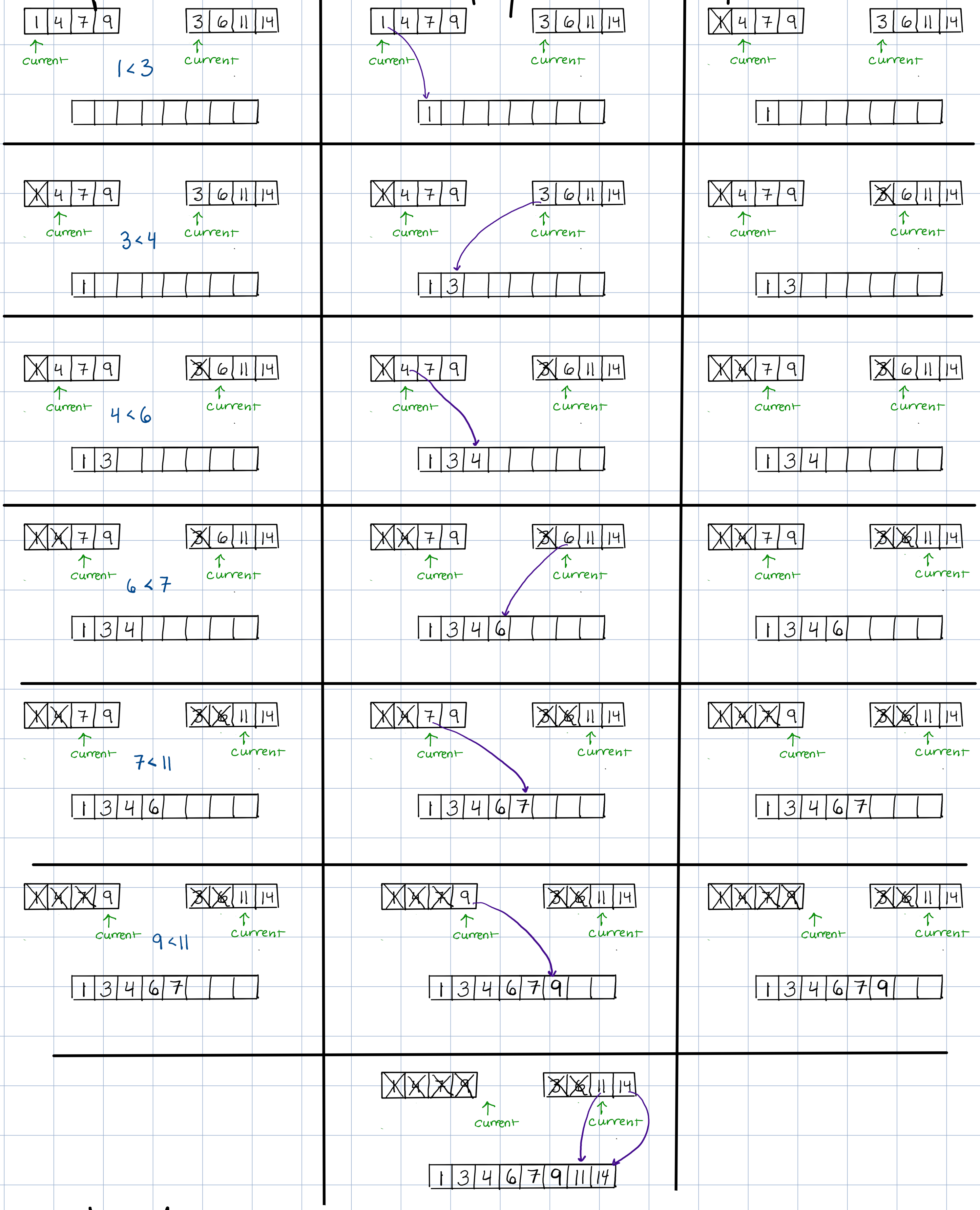
Basic Building Block: merging sorted lists



1) Find current smallest, move to final list
2) increment current

Repeat until one list is gone → then just place all others into final list

Comparison | Copy | Update Current



Note:1 when reaching 11,14 left can just add them directly

Exercise| Build a worst case (the most comparisons) of merging two sorted lists of length 4.

(How many comparisons for two length $n/2$ lists?)

$\boxed{1\ \ 3\ \ 5\ \ 7}$ $\quad$ $\boxed{2\ \ 4\ \ 6\ \ 8}$

$1<2$, $3>2$, $3<4$, $5>4$, $5<6$, $7>6$, $7<8$, 8 for free

7 comparisons!

$n-1$ comparisons

Worst Case: merging two sorted lists

Every comparison moves one element to final list

If one list runs out no comparisons needed for elements left in other list!

Worst case: $N/2$ length (N total)

$\boxed{N-1 \text{ comparisons}}$

Last element we get for "free"!

But let's just say $O(N)$ comparisons to merge two $N/2$ length sorted lists!

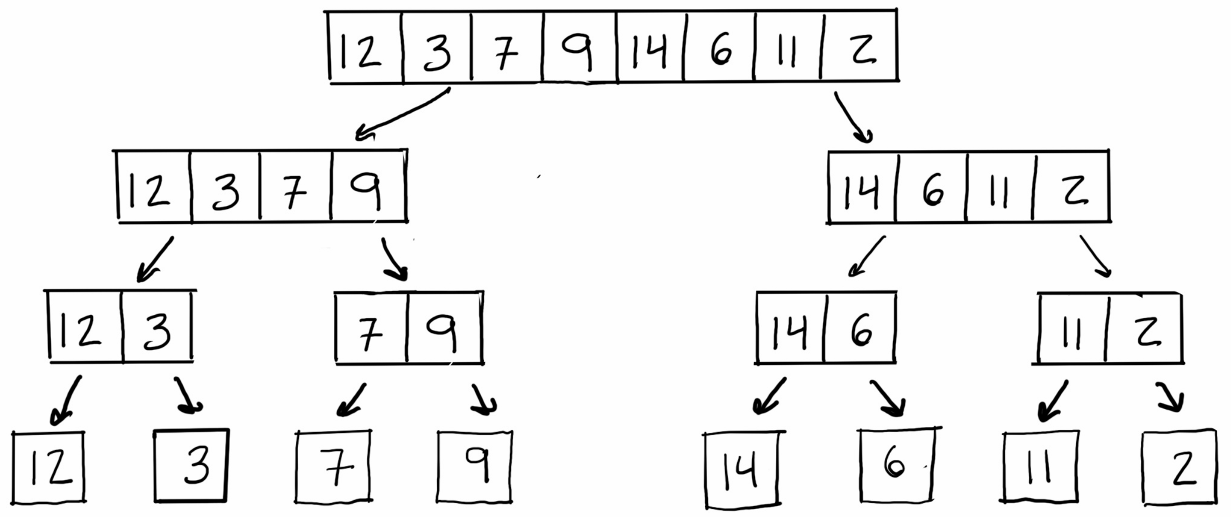
How do we use this to sort?

Remember from insertion sort, a single element is sorted!

12	3	7	9	14	6	11	2

Approach: divide list in half until all length 1 (they are sorted!)

Then merge sorted lists back together with our earlier technique



Observations

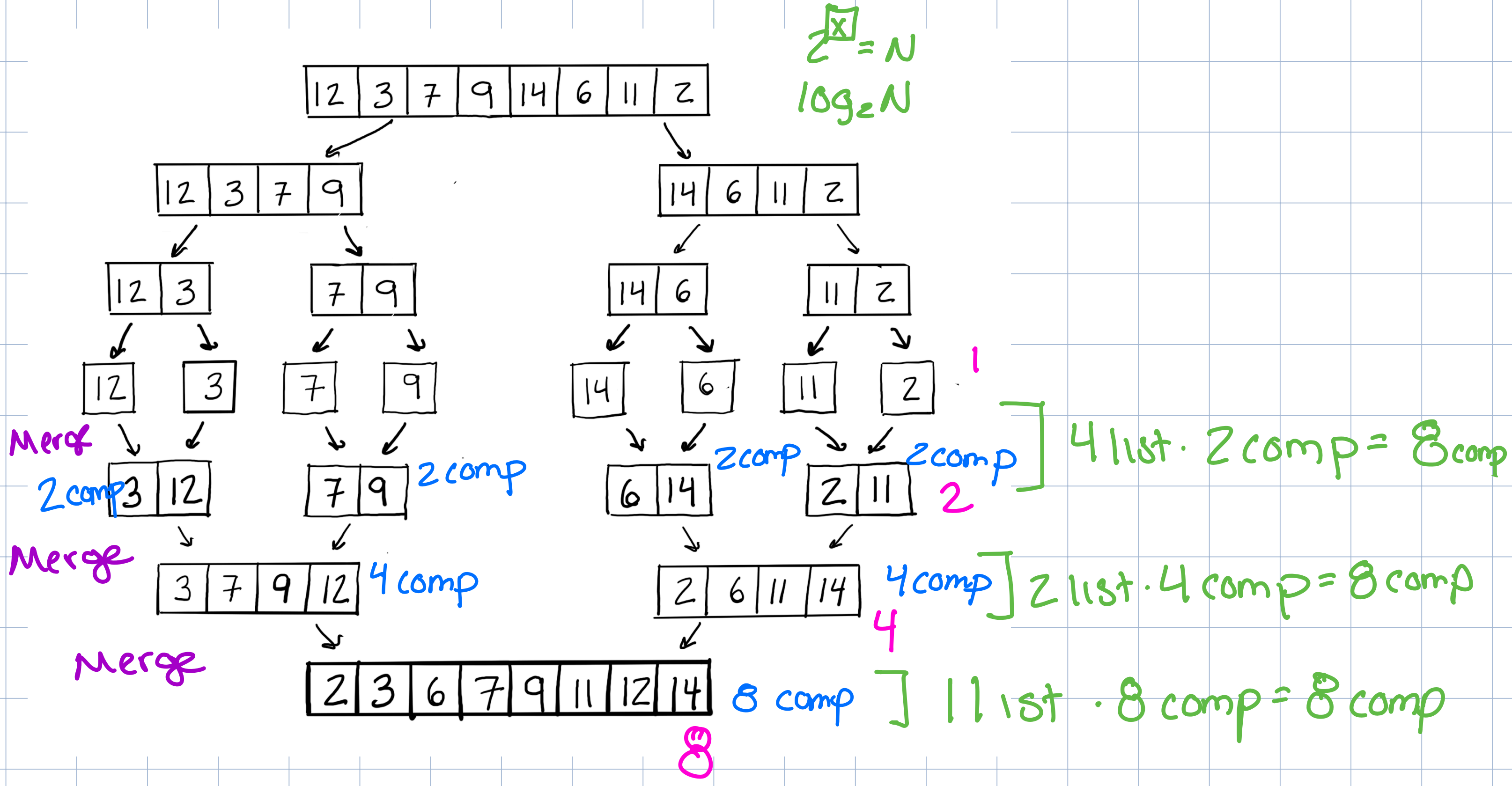
1) This is a type of algorithm knowns as divide and conquer; divide until sub problems are easily solved

(Binary Search!)

2) Divide steps require no comparisons

3) Each merge step in worst case requires O(N) comparisons

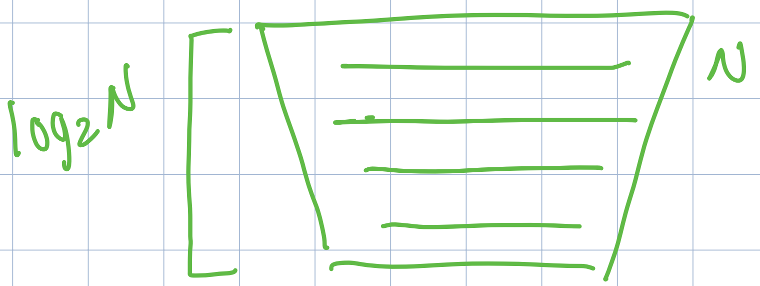
(see next page)



3 levels merging · 8 comparisons per level = $\boxed{24 \text{ comp}}$

For general N:

$\log_2 N$ levels merging · N comparisons per level = $N \log N$ comp



Mergesort has runtime of:
$\Theta(n \log n)$

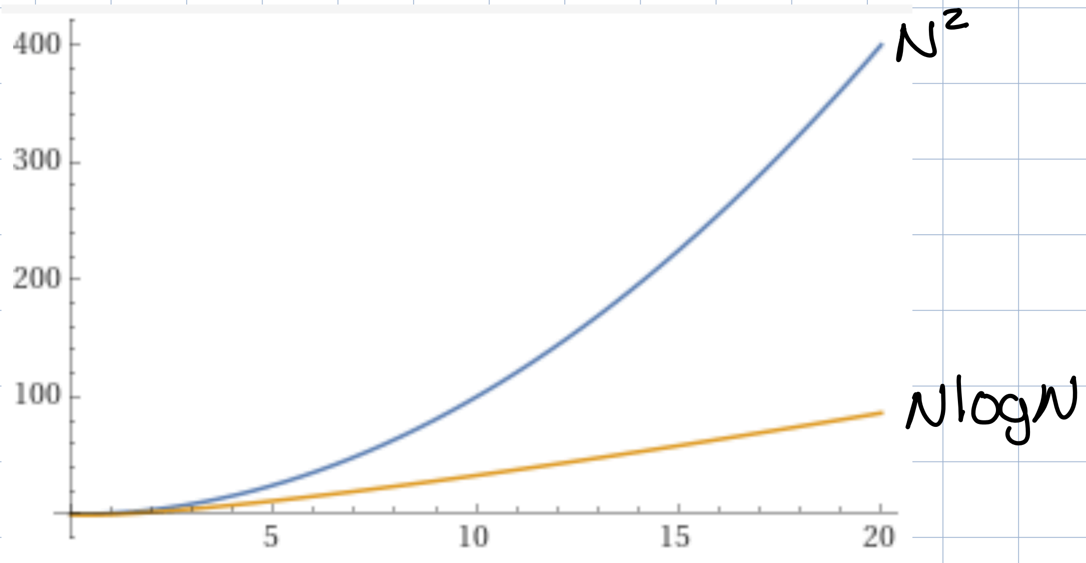
Comparing sorting algorithms

Insertion Sort:

$O(N^2)$

MergeSort:

$O(N\log N)$

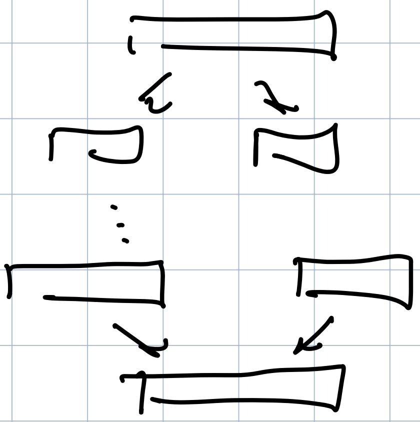


Recurrence Relations

Another way of analyzing mergesort's runtime

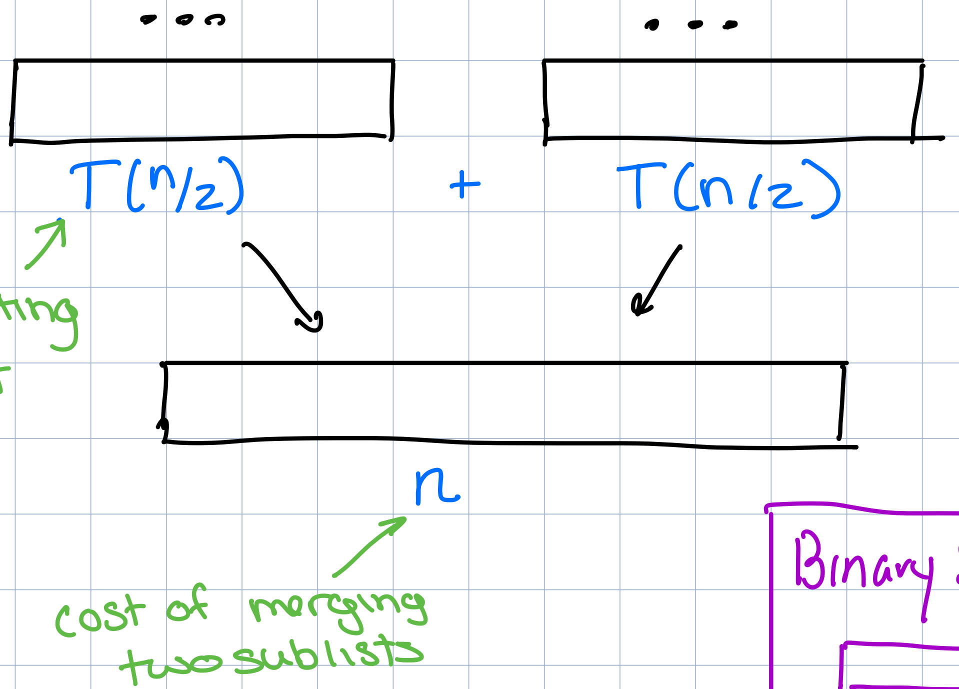
Why bother? Because other divide and conquer algorithms are harder to think about than mergesort

Intuition:



Call $T(n)$ the cost of - doing merge sort on list of size n. Can break down this cost...

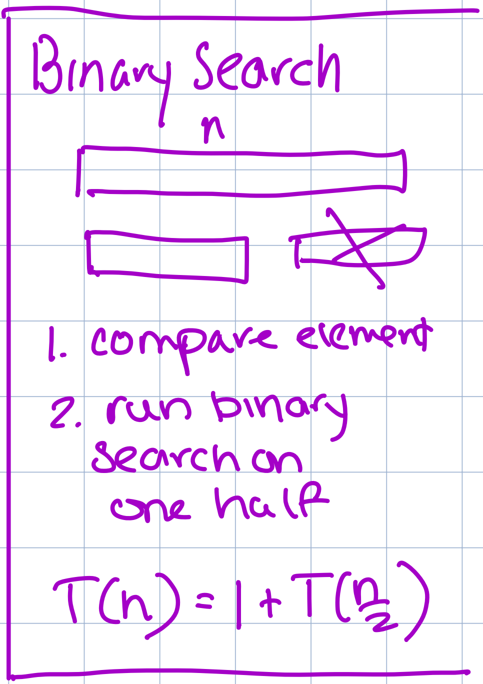
Breakdown



$$T(n) = T(n/2) + T(n/2) + n$$

~or~

$$\boxed{T(n) = 2T(n/2) + n}$$



1. compare element
2. run binary search on one half

$$T(n) = 1 + T(\tfrac{n}{2})$$

This is a recurrance relation, each item is expressed in terms of previous item

$$T(n) = 2T(n/2) + n$$

This doesn't exactly give us a runtime though.

How can we get an expression w/ $T(\cdot)$ only on one side? (closed form)

Substitution Method

aka... expand a few layers and notice a pattern

Expanding one layer:

$T(n) = 2T(n/2) + n$

can substitute in $T(n/2)$:

$T(n) = 2\left(2T(n/4) + \frac{n}{2}\right) + n$

$T(n) = 2^2 T(n/4) + 2n$

What is $T(n/2)$:

For clarity call $\square = n/2$

$T(\square) = 2T(\square/2) + \square$

Now if we plug $n/2$ for $\square$

$T(n/2) = 2T(\cancel{n/2}/2) + \frac{n}{2}$

$T(n/2) = 2T(n/4) + \frac{n}{2}$

Expanding Second layer!

$T(n) = 2^2 T(n/4) + 2n$

$T(n) = 2^2\left(2T(n/8) + n/4\right) + 2n$

$T(n) = 2^3 T(n/8) + 3n$

What is $T(n/4)$?

$T(n/4) = 2T(n/8) + n/4$

Now let's see if we put our expressions side by side if we spot a pattern...

$T(n) = 2T(n/2) + n$ one expansion

$T(n) = 2^2 T(n/4) + 2n$ two expansions

$T(n) = 2^3 T(n/8) + 3n$ three expansions

More formally:

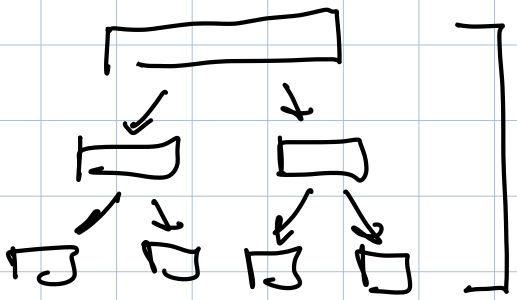
$T(n) = 2^1 T(n/2) + 1n$ one expansion $k=1$

$T(n) = 2^2 T(n/4) + 2n$ two expansions $k=2$

$T(n) = 2^3 T(n/8) + 3n$ three expansions $k=3$

$= 2^k T(n/2^k) + kn$

Okay but what is k?
k is the number of expansions we can do before we reach the "bottom" or base case



In merge sort we can only divide list until they are 1 element!

What is our base case? And how much work do we do there?

In Mergesort, our smallest list is size 1 and it is already sorted so we do no work e.g.

$$T(1) = 0$$

Solving for k:

Once we have our base case we can find how many substitutions we can make before reaching it

$$T(n) = 2^k T(\underbrace{n/2^k}) + kn$$

aka what value of k makes this = 1

Thing in T(□)

$$\overbrace{n}\over 2^k} = \underset{\uparrow}{1} \Rightarrow n = 2^k \Rightarrow k = \log_2 n$$

base case value

So lets eliminate k in our previous expression:

$$\begin{aligned} T(n) &= 2^k T(n/2^k) + kn \\ &= 2^{\log_2 n} T(n/2^{\log_2 n}) + n\log_2 n \\ &= n \cdot T(1) + n\log_2 n \\ &= n\log_2 n \end{aligned}$$

we know T(1)=0 !

same as before!

Condensed solution for reference:

$$T(n) = 2T(n/2) + n \qquad T(n/2) = 2T(n/4) + \frac{n}{2}$$

$$T(n) = 2(2T(n/4) + \frac{n}{2}) + n$$

$$= 2^2 T(n/4) + 2n \qquad T(n/4) = 2T(n/8) + n/4$$

$$T(n) = 2^2(2T(n/8) + n/4) + 2n$$

$$= 2^3 T(n/8) + 3n$$

After k substitutions:

$$T(n) = 2^k T(n/2^k) + kn$$

Solving for k when $T(1) = 0$:

$$\frac{n}{2^k} = 1 \Rightarrow n = 2^k \Rightarrow k = \log_2 n$$

Substituting k back in:

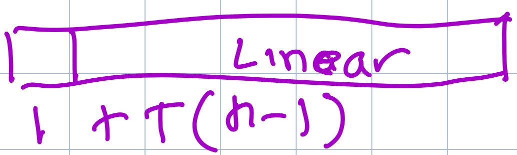
$$T(n) = 2^k T(n/2^k) + kn$$

$$= 2^{\log_2 n} T(n/2^{\log_2 n}) + n\log_2 n$$

$$= n \cdot T(1) + n\log_2 n$$

$$= n\log_2 n$$

$$\boxed{T(n) = n\log_2 n}$$



Exercise:

1) $T(n) = T(n-1) + 1$, $T(1) = 1$ $\boxed{T(n) = n}$

$T(n) = T(n-1) + 1$ $T(n-1) = T(n-2) + 1$

$T(n) = (T(n-2) + 1) + 1$

$= T(n-2) + 2$ $T(n-2) = T(n-3) + 1$

$T(n) = (T(n-3) + 1) + 2$

$T(n-3) + 3$

General form:

$T(n) = T(n-k) + k$

Solve for k:

$n - k = 1 \Rightarrow k = n - 1$

Substitute k back in

$T(n) = T(n-(n-1)) + (n-1)$

$= T(n-n+1) + n - 1$

$= T(1) + n - 1$

$= 1 + n - 1$

$= n$

This is recurrance for linear search.

2) $T(n) = T(n-3) + 4$, $T(1) = 1$ $T(n) = \frac{4n-1}{3}$

$T(n) = 1(n-3)+4$ $T(n-3) = T(n-6) + 4$

$T(n) = T(n-6) + 4 + 4$

$= T(n-2\cdot3) + 2\cdot4$

$T(n) = T(n-6) + 2\cdot4$ $T(n-6) = T(n-9) + 4$

$= (T(n-9) + 4) + 2\cdot4$

$T(n) = T(n-3k) + 4k$

Solve for k:

$n - 3k = 1 \Rightarrow 3k = n-1 \Rightarrow k = \frac{n-1}{3}$

Plug back in

$$T(n) = T\left(n - 3\left(\frac{n-1}{3}\right)\right) + 4\left(\frac{n-1}{3}\right)$$

$$= T(n - n + 1) + 4\left(\frac{n-1}{3}\right)$$

$$= 1 + 4\left(\frac{n-1}{3}\right)$$

$$= \frac{4n-4}{3} + \frac{3}{3}$$

$$= \boxed{\frac{4n-1}{3}}$$

3) $T(n) = 7T(n-2)$ $\quad T(0)=1$ $\quad T(n) = 7^{n/2}$

$T(n) = 7T(n-2)$

$= 7 \cdot (7T(n-4))$

$= 7^2 T(n-4)$

$= 7^2(7T(n-6))$

$= 7^3 T(n-6)$

$T(n-2) = 7T(n-4)$

$T(n-4) = 7T(n-6)$

$T(n) = 7^k T(n-2k)$

Solve for k:

$n - 2k = 0 \Rightarrow k = n/2$

Plug back in:

$T(n) = 7^{n/2} T(n - 2(\frac{n}{2})$

$7^{n/2} \cdot 1$

$\boxed{= 7^{n/2}}$