Agenda

Professor Hamlin
Day 10

1) Admin
   - Practice exam
   - Exam instructions
   - Leave answers in factorial form (HW)
2) Review
3) Combinations (order doesn't matter)
   - left over principle
4) Balls & Bins

Review

	Can reuse options (e.g. passwords)	Can't reuse options (e.g. people in photo)
Permutations (order matters)	n^k	$P(n,k) = \frac{n!}{(n-k)!}$

Counting stragetes - count by partition, complement, simplification

Exercise: 1) How many orders can I pet 5 cats at a cat cafe, if I can pet a cat more than once? with 10 cats total

10^5

2) Same question, but now I can't pet a cat more than once?

$P(10,5) = \frac{10!}{5!}$

Combination - (order does not matter)

Grab a fist of two candies from the halloween candy bowl (Jolly Rancher, Reeses Cup, Milky way)
J R M

How many different combinations?

(J, R) (J, M) (R, M)
(R, J) (M, J) (M, R)

In ordered pairs this is $P(3,2) = \frac{3!}{1!} = 6$

But I'm grabbing a fistful at a time → it doesn't matter what order I grab them in!

the same when order doesn't matter

(J, R) (J, M) (R, M)
(R, J) (M, J) (M, R)

3 ways unordered

We've overcounted, we have 2x the count.
OR there are 2! = 2 ways of ordering 2 candies

$$\frac{6}{2!} = \frac{6}{2} = \boxed{3}$$

Informally combinations are:

$$\text{Ways of choosing 2 from 3 (order not matter)} = \frac{\text{Ways of ordering 2 from 3 } (\cancel{\text{order matters}})}{\text{ways of ordering 2 } (\cancel{\text{order matters}})}$$

Formally: choosing k from n items, order doesn't matter

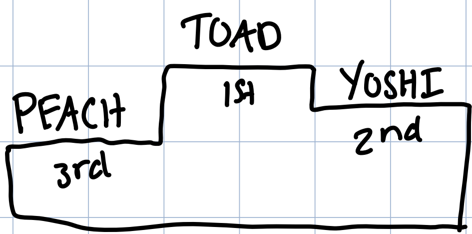
$$\boxed{C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}}$$

"n choose k"

	Can reuse options (e.g passwords)	Can't reuse options (e.g. people in photo)
Permutations	n^k	$P(n,k)$ $\frac{n!}{(n-k)!}$
Combinations	? come back to this	$C(n,k)$ $\binom{n}{k} = \frac{n!}{(n-k)k!}$

Exercise 1) How many ways can 8 mariokart racers form the top 3. The order of 1st, 2nd, & 3rd matters

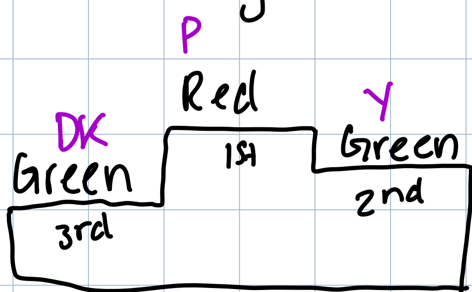
order matters, no repeats



$P(8,3) = \boxed{\frac{8!}{5!}}$

2) We now have 4 teams in mariokart, with 3 members each. How many different team orderings can we have for 1st, 2nd, 3rd?

order matters, can repeat



$n = 4 \qquad k = 3$

$\boxed{4^3}$

3) How many unique hands of 5 cards can be made from standard 52 card deck? (Hands are unordered)

unordered, no repeats

$n = 52 \qquad k = 5$

$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \boxed{\frac{52!}{5!\,47!}}$

4) How many ways can one select the "remaining" 47 cards from the problem above?

$n = 52 \qquad k = 47$

$\binom{52}{47} = \frac{52!}{47!\,(52-47)!} = \boxed{\frac{52!}{47!\,5!}}$

This is equal!

Leftover principle

How many ways can I choose all but 10 students to take out for ice cream in class of size n?

$$\binom{n}{10} = \boxed{\frac{n!}{(n-10)!\,10!}}$$

How about the n-10 to take out for ice cream?

$$\binom{n}{n-10} = \frac{n!}{(n-(n-10))!\,(n-10)!}$$

$\uparrow$ k

$$\frac{n!}{(\cancel{n}-\cancel{n}+10)!\,(n-10)!} = \boxed{\frac{n!}{10!\,(n-10)!}}$$

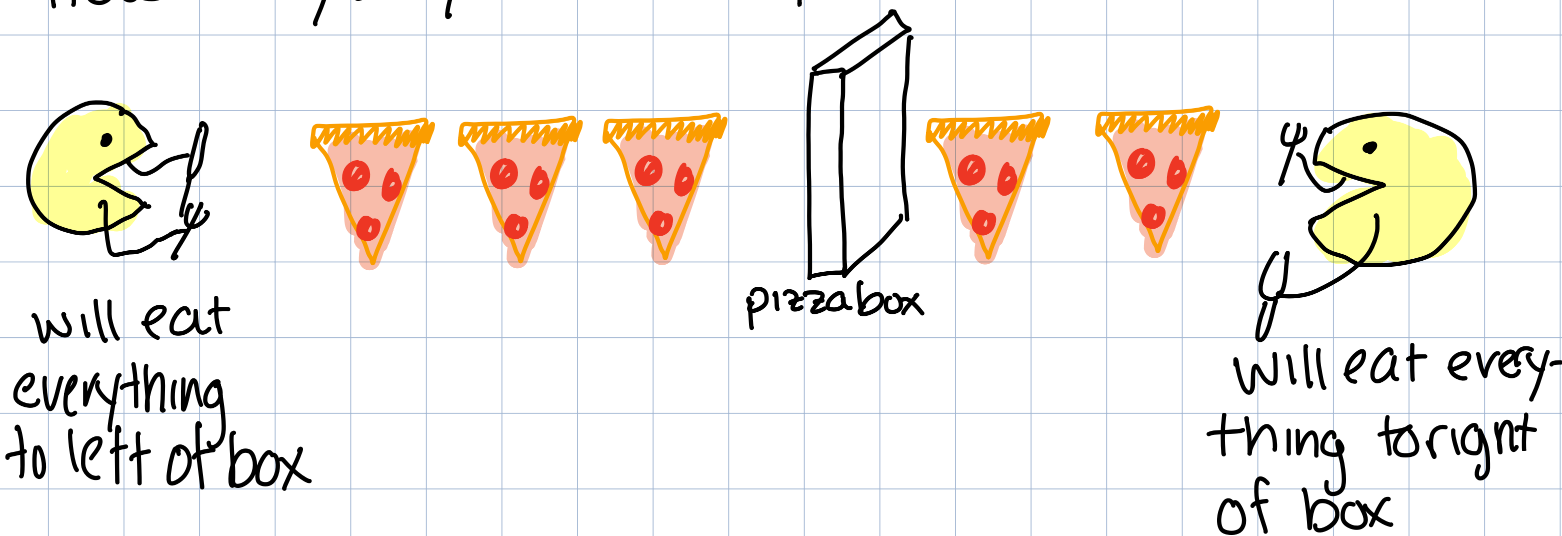
> It doesn't matter who we choose to include or exclude, it is the same # of combinations

	Can reuse options (e.g passwords)	Can't reuse options (e.g. people in photo)
Permutations	n^k	$\frac{n!}{(n-k)!}$
Combinations	? Come back to this	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

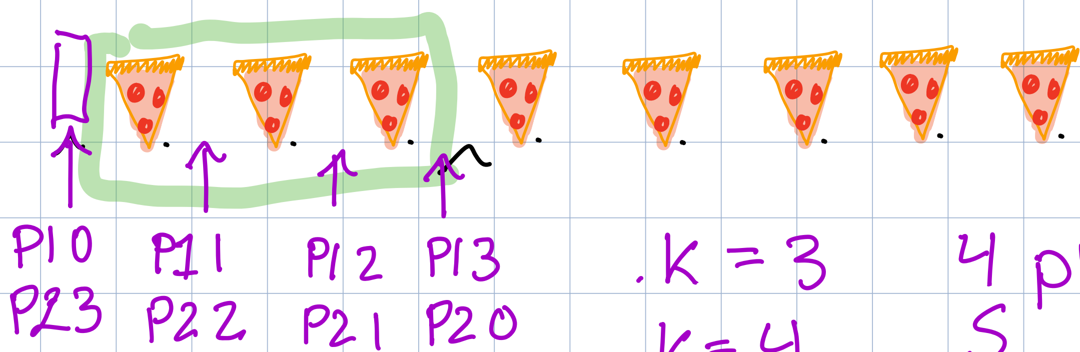
Counting partitions of identical objects

(Balls & Bins or Stars & Bars)

How many ways can two people split k slices of pizza?



1. Pizza slices are identical so they can be in any order
2. What varies is the number of pizza each eats, i.e. where we put the barrier



$k=3$	4 places for boun
$k=4$	5
$k=5$	6

$\binom{3+1}{1}$

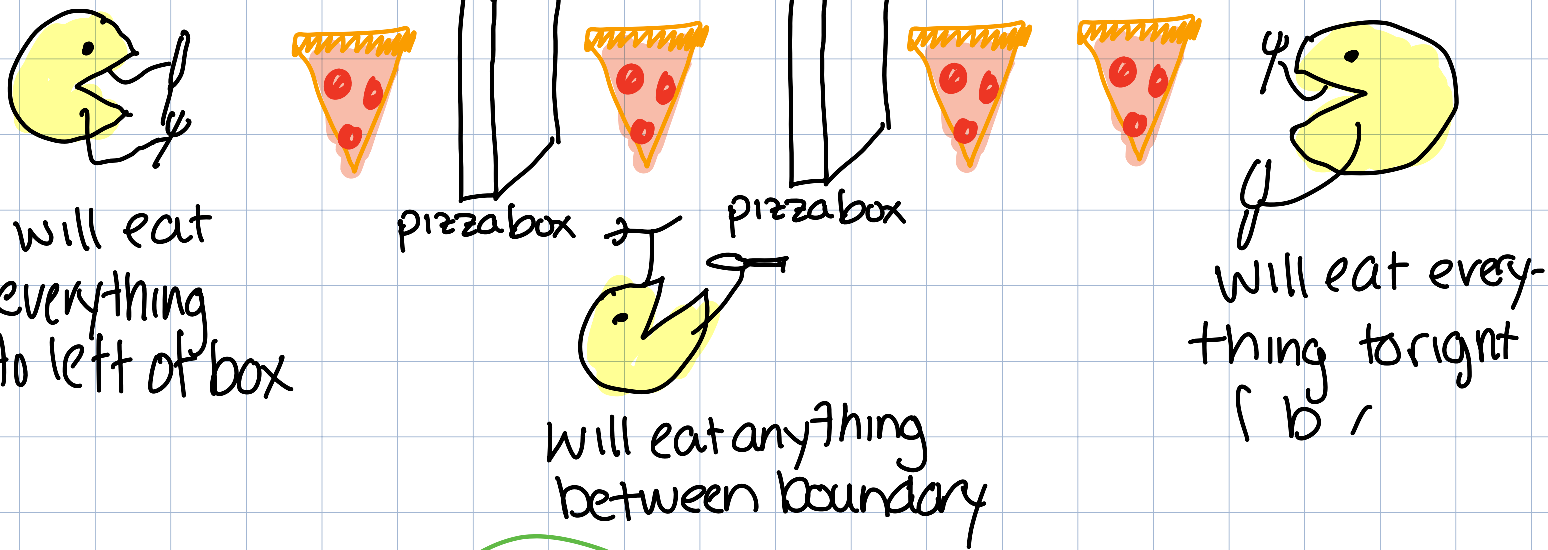
$\binom{4}{1} = \frac{4!}{1!\,3!} = \boxed{4}$

$\binom{k+1}{1}$ ways

k slices of pizza

1 boundary

Three people?



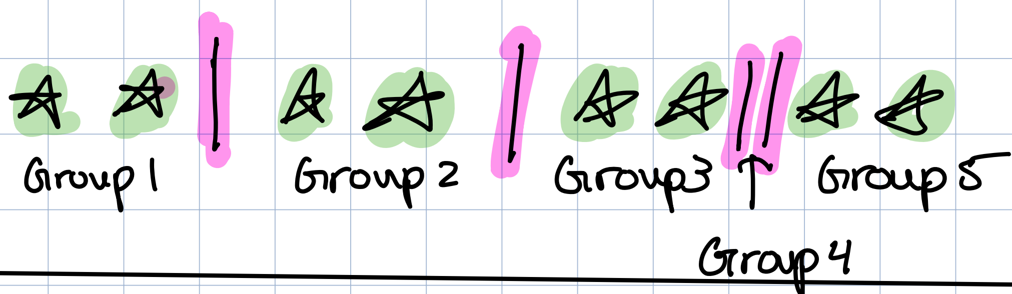
$$\binom{k+2}{2} \text{ ways}$$

\# of slices

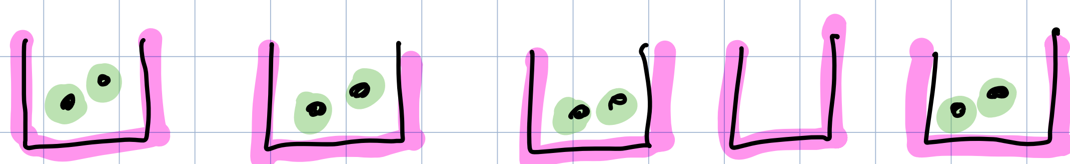
\# of boundaries

Formally if we have n piles and k items
(Bins) & (Balls)
(Bars) & (stars)

stars & bars



balls & bins



Need n-1 boundaries for n piles

$$\binom{k+n-1}{n-1} \text{ ways}$$

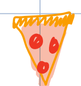
Putting it together	Can reuse options (e.g. passwords)	Can't reuse options (e.g. people in photo)
Permutations (order matters)	n^k	$\frac{n!}{(n-k)!}$
Combinations (order doesn't matter)	$\binom{k+n-1}{n-1}$	$\binom{n}{k} = \frac{n!}{(n-k)!k!}$

Example: 10 candy pieces, 3 tricker treaters

\# of ways divide candy:  $n=3$ $K=10$

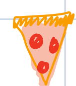
$$\binom{10+3-1}{3-1} = \binom{12}{2}$$

Warning: two equivalent formulas for balls & bins

K item 
N groups 

$$\binom{K+N-1}{N-1}$$

(What we used)

K item 
N groups 

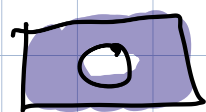
$$\binom{N+K-1}{K}$$

Equivalent b/c leftover principle

Counting in summary:

- Sum rule: an ~or~ of options
  - if disjoint $|A \cup B| = |A| + |B|$
  - if not, use PIE: $|A \cup B| = |A| + |B| - |A \cap B|$

- Product rule: an ~and~ of options
  $|A \times B| = |A| * |B|$

- Counting stragegies
  - Count-by-partition: break into disjoint subsets, combine using sum rule
  - Count-by-complement: count items not of interest  $|U - I| = |U| - |I|$
  - Count-by-simplification: look for simplier, equivalent problems

Advice:

1. Clearly document approach (easier to find mistakes)
2. If stuck, review counting rules / stragegies
   try solving simplier sub problem
   determine if order matters & are repeats allowed

1) How many passwords of length 5 can be made from vowels (upper and lowercase)?

AEIOU

order matters 10

can repeat

$n^k = \boxed{10^5}$

2) How many ways can I select 10 students in this room to give a million extra credit points to?

(125 students)

order doesn't matter

cannot repeat

$\binom{n}{k} = \binom{125}{10} = \boxed{\frac{125!}{115!\,10!}}$

3) 10 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

order matters, can't repeat

$n = 10 \qquad k = 3$

$P(10,3) = \frac{10!}{7!}$

4) How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

order doesn't matter,

can repeat pizza types

Balls = Pizza = 14

Bins = flavor = 3

$\binom{14+3-1}{3-1} = \binom{16}{14}$

$\boxed{\frac{16!}{14!\,2!}}$

6) I've got 3 pairs of pants, 2 shirts and 5 hats. How many outfits (pants, shirt & hat) can I wear if I won't wear one pair of pants with either 1 shirt or 1 hat?

i.e. won't wear pant A w/ shirt B and pant A w/ hat C

complement then partition $|U| - |I|$

$|U|$ = total unrestricted # of outfits

$|I|$ = shirt B w/ pant A ~or~ pant A w/ hat C

$\underset{P}{3} \cdot \underset{S}{2} \cdot \underset{H}{5} = 30$

$\underset{P}{1} \ \underset{S}{1} \ \underset{H}{5} = 5$ $\quad$ $\underset{P}{1} \ \underset{S}{2} \ \underset{H}{1} = 2$

$|U| - (|A_p \text{ w/ } B_s| + |A_p \text{ w/ } C_h| - 1)$

$30 - 2 - 5 + 1 = \boxed{24}$ $\quad$ $\underset{P}{A} \ \underset{S}{B} \ \underset{H}{C}$

7) 5 countries each have 2 woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

No nation w/ 2 swimmers

$P(5,3)$

nation repeated

			A B
A	A	B	$5 \cdot 4$
A	B	A	$5 \cdot 4$
B	A	A	$5 \cdot 4$

$3 \cdot 5 \cdot 4$

$\boxed{\frac{5!}{3!} + 3 \cdot 5 \cdot 4}$

8) ~~[illegible]~~ 5 countries each have ~~[illegible]~~ 3 woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged?
(e.g. in tokyo 2020 it was Australia, Hong Kong (China) & Canada)

No nation repeated	2 repeated	3 times
" "	" "	A A A 5

$$\boxed{\frac{5!}{3!} + 5 \cdot 3 \cdot 4 + 5}$$

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

(++) redo the pizza problem, relaxing our assumption that the whole pizza may only be of one type. Instead, assume each half of the pizza may only be of one type.

Bins are still pizza types, we just have more options

CP CV PV
CC VV PP

6 bins, 14 pizza $\binom{14+6-1}{6-1} = \binom{19}{5}$