Professor Hamlin
Day 18

Agenda

1) admin → Exam 2 this Friday (this material not on exam)
→ no quiz this week
2) Review
3) Series & Sequences
   - Arithmetic
   - Geometric
   - Quadratic

   1) Recognize
   2) i-th term
   3) Partial sum

Review

Weak induction on inequalities

1: $x < y$ $x + c < y + c$

2: $x < y$ $x \cdot c < y \cdot c$ $c > 0$

3: $x < y$ $x \cdot c > y \cdot c$ $c < 0$

4: $x < y$ $w < z$ $x + w < y + z$

5: $x < y$ $z < x$ $z < y$

Strong induction stairs vs ladder

assume $S(1), S(2), \ldots S(n)$ vs $S(n)$

Exercise

Which are valid manipulations of inequalities (~~and what moves~~)

1) $x + 10 < z \Rightarrow x < z$ $x > 0,\ z > 0$

Valid

2) $x < y,\ y < z \Rightarrow x < z$

Valid

3) $x - 10 < y \Rightarrow 10 - x < -y$ $10 - x > -y$

invalid

Last thing from last class: summation

last value of k

$$\sum_{k=0}^{4} 1+2^k = \begin{array}{l} (1+2^0) \\ +(1+2^1) \\ +(1+2^2) \\ +(1+2^3) \\ +(1+2^4) \end{array} = \begin{array}{l} 1 \\ +3 \\ +5 \\ +9 \\ +17 \end{array} = 35$$

1st value of k

k is the value always increasing by 1

$\boxed{\text{Things to sum can come in some common formats}}$

Think about:

1, 3, 5, 7, 9,... vs 2, 6, 10, 14, ...

1, 2, 4, 8, 16, ... vs 1, 3, 9, 27, ...

Feel similar but something like...

2, 4, 6, 8, 10... vs 1, 4, 9, 25, 36

Grows at a very different rate! We can capture this idea formally. But first some vocab.

Sequence: an ordered list of objects

e.g. 1, 2, 3, 4, 5, 6 ....

Series: sum of an infinite sequence

e.g $1+2+3+4\ldots = \sum_{k=1}^{\infty} k$

Term: individual object in series/sequence

e.g $1, 2, 3, 4\ldots$ 2 is 2^{nd} term

Partial Sum: sum of just part of a series

e.g $1+2+3+4 = \sum_{k=1}^{4} k = 10$

First type of Sequence: Arithmetic

first difference

$$\underbrace{10, 12}_{+2}, \underbrace{14}_{+2}, \underbrace{16}_{+2}, \underbrace{18}_{+2}, \underbrace{20}_{+2}, \ldots$$

$$\underbrace{18, 12}_{-6}, \underbrace{6}_{-6}, \underbrace{0}_{-6}, \underbrace{-6}_{-6}, \underbrace{-12}_{-6}, \ldots$$

What do these two sequences have in common?

The difference (next term - previous term) is constant

How to tell if sequence is Arithmetic? Check if difference between terms is constant

Arithmetic Series / Partial Sum

$$10 + 12 + 14 + 16 + 18 + 20 + \ldots = \sum_{k=0}^{\infty} 10 + 2k$$

$10+2\cdot 0$ $10+2\cdot 1$ $10+2\cdot 2$

General form is:

$$\sum_{k=0}^{\infty} a_0 + dk$$

1st term in sequence

difference between terms

<u>Example</u> $5 + 2 + -1 + -4 \ldots$

$a_0 = 5$

$d = 2 - 5 = -3$

$$\sum_{k=0}^{\infty} 5 - 3k$$

Second type of sequence: Geometric

$\frac{1}{2}, 1, 2, 4, 8, 16, \ldots$

$\times 2$ $\times 2$ $\times 2$ $\times 2$ $\times 2$

$100, -10, 1, -\frac{1}{10}, \frac{1}{100}, \ldots$

$\times -\frac{1}{10}$ $\times -\frac{1}{10}$ $\times -\frac{1}{10}$ $\times -\frac{1}{10}$

What do these two sequences have in common?

The ratio (next term/current term) is constant

$$\boxed{\text{How to tell sequence is Geometric? Divide next term by previous term and result is constant}}$$

Example: 100, -10, 1, ...

ratio: $\frac{-10}{100} = \frac{-1}{10}$ ratio: $\frac{1}{-10} = \frac{-1}{10}$

Geometric Series / Partial Sum

$$\frac{1}{2} + 1 + 2 + 4 + 8 + \ldots = \sum_{k=0}^{\infty} \frac{1}{2} \cdot 2^k$$

$\frac{1}{2} \cdot 2^0$ $\frac{1}{2} \cdot 2^1$ $\frac{1}{2} \cdot 2^2$

General Form is:

$$\sum_{k=0}^{\infty} a_0 r^k$$

starting term

ratio of $\frac{\text{next term}}{\text{previous term}}$

Example/ $18 + 6 + 2 + \frac{2}{3} + \frac{2}{9}, \ldots$

$a_0 = 18$

$r = \frac{6}{18} = \frac{1}{3}$

$$\sum_{k=0}^{\infty} 18 \cdot \left(\frac{1}{3}\right)^k$$

Third type of sequence is: Quadratic

Harder to see than arithmetic/geometric

1, 3, 7, 13, 21, 31

Arithmetic? +2 +4 +6 +8 +10 Nope!

Geometric? 1/3 3/7 7/13 13/21 21/31 Nope!

What they actually look like is....

$$a_n = an^2 + bn + c$$

n-th term starting w/ n=0

constant values

Example; $a = 1$, $b = 0$, $c = 0$ e.g. $a_n = 1n^2 + 0 \cdot n + 0$

N=0 N=1 N=2

0, 1, 4, 9, 16, 25,

$1 \cdot 0^2 + 0 \cdot 0$ $1 \cdot 1^2 + 0 \cdot 1 + 0$ $1 \cdot 2^2 + 0 \cdot 2 + 0$

How do we identify Quadratic sequence/series?

1, 3, 7, 13, 21, 31

first diff. +2 +4 +6 +8 +10

Second diff: +2 +2 +2 +2

$\boxed{\text{The second difference is constant}}$

Exercise 1 Identify arithmetic, geometric, quadratic or none. If arithmetic or geometric write in sum notation

1) 6, 15, 28, 45, 66, 91 — Quadratic

9 13 17 21

4 4 4

2) 1, −4, 16, −64, 256, ... — Geometric

x−4 x−4 x−4

$$\sum_{k=0}^{\infty} 1 \cdot (-4)^k$$

3) 4, 7, 10, 13, 16, 19, ... — Arithmetic

+3 +3 +3 +3

$$\sum_{k=0}^{\infty} 4 + 3k$$

4) 2, 7, 11, 42, −4 — None

5 4 31 −46

1 27

$^2/_7$ $^7/_{11}$

Coming back to getting a, b, c for Quadratic from the sequence

$\sum ak^2 + bk + c$

$6 + 15 + 28 + 66 + 91 + \ldots$

$k=0$ $\quad$ $k=1$ $\quad$ $k=2$

system of equations
$$\begin{cases} 6 = a\cdot 0^2 + b\cdot 0 + c \Rightarrow c = 6 \\ 15 = a\cdot 1^2 + b\cdot 1 + c \\ 28 = a\cdot 2^2 + b\cdot 2 + c \end{cases}$$

Solving system of equations:

$$6 = c$$
$$15 = a + b + c$$
$$28 = 4a + 2b + c$$

Substitute in c and simplify

$$15 = a + b + 6 \Rightarrow 9 = a + b$$
$$28 = 4a + 2b + 6 \Rightarrow 22 = 4a + 2b$$

Solve for a / b to substitute into other equation:

$$9 = a + b \Rightarrow b = 9 - a$$

Substitute into other eqn and simplify

$$22 = 4a + 2(9 - a)$$

$$22 = 4a + 18 - 2a$$
$$4 = 2a$$
$$a = 2$$

Substitute into original eqn:

$$b = 9 - a \Rightarrow b = 9 - 2 \Rightarrow b = 7$$

Thus, $a = 2$, $b = 7$, $c = 6$

Checking work: $a_n = 2n^2 + 7n + 6$ $\quad n \geq 0$

$$6 + 15 + 28 + 66 + 91 + \dots$$

$6 = 2 \cdot 0^2 + 7 \cdot 0 + 6$ ✓
$15 = 2 \cdot 1^2 + 7 \cdot 1 + 6$ ✓
$28 = 2 \cdot 2^2 + 7 \cdot 2 + 6$ ✓

> Note we want to start Quadratic series at $k=0$ as it makes solving this easier

Exercise Find a, b, c for

$$\underset{k=0}{1} + \underset{k=1}{3} + \underset{k=2}{7} + 13 + 21 + 31 + \dots = \sum_{k=0}^{\infty} ak^2 + bk + c$$

$a \cdot 0^2 + b \cdot 0 + c = 1 \Rightarrow c = 1$
$a \cdot 1^2 + b \cdot 1 + c = 3$
$a \cdot 2^2 + b \cdot 2 + c = 7$

Thus

$$a+b=2 \quad \Rightarrow \quad b=2-a$$
$$4a+2b=6$$

Thus $4a+2(2-a)=6$

$$2a=2$$
$$a=1$$

thus $b=2-1=1$

Up next: Partial Sums (Arithmetic/Geometric)

Arithmetic:

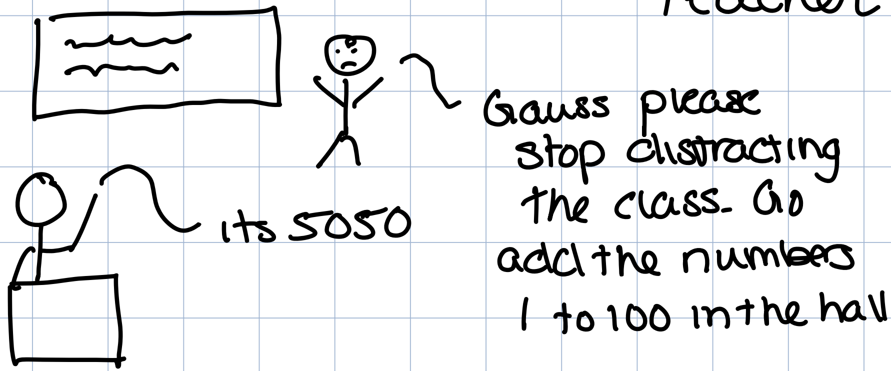
$$0+1+2+3+4 = \sum_{k=0}^{4} k = ? \qquad 10$$

Geometric:

$$1+2+4+8+16 = \sum_{k=0}^{4} 2^k = ? \qquad 31$$

Faster way than just summing everything up?
Yes!

Apocryphal tale: Gauss and a frustrated teacher



$$1 + 2 + 3 + \dots \quad \dots + 98 + 99 + 100$$

101

101

101

50 sums of 101 $\Rightarrow$ $50 \cdot 101 = 5050$

General Form:

first term

last term

$$\sum_{k=0}^{N} a_0 + dk = (a_0 + a_N)\cdot\left(\frac{N+1}{2}\right)$$

number of pairs

Example:

$$\begin{array}{ccccc} N=0 & 1 & 2 & 3 & 4 \\ 1+ & 2+ & 3+ & 4+ & 5 \end{array}$$

$$\boxed{\sum_{k=0}^{4} 1+k}$$

$a_0 = 1$

$a_N = 5$

$N = 4$

careful, 5 terms but we start at 0 so $N=4$

$$(1+5)\left(\frac{4+1}{2}\right) = \boxed{15}$$

Geometric Series Partial Sum

This can be a bit unintuitive for how we get this equation. Humor me for a moment

$$S = \sum_{k=0}^{N} ar^k = a + ar + ar^2 + \ldots + ar^{N-1} + ar^N$$

Partial Sum we want

So let's compute $r \cdot S$, for fun....

$$r \cdot S = ar + ar^2 + ar^3 + \ldots + ar^N + ar^{N+1}$$

So consider the following:

$$S - rS = (a + \cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{N-1}} + \cancel{ar^N})$$
$$- (\cancel{ar} + \cancel{ar^2} + \ldots + \cancel{ar^{N-1}} + \cancel{ar^N} + ar^{N+1})$$

all of these terms cancel out

leaving $S - rS = a - ar^{N+1}$

Remember S is what we want to compute so we solve for S

$$\frac{S(1-r) = a - ar^{N+1}}{1-r}$$

Thus

$$\boxed{\sum_{k=0}^{\infty} a_0 r^k = S = \frac{a_0(1-r^{N+1})}{1-r}}$$

Example $1+2+4+8+16$

$$\sum_{k=0}^{4} 1 \cdot 2^k \qquad S = \frac{1(1-2^{4+1})}{1-2} = \frac{-31}{-1} = \boxed{31}$$

remember even though 5 terms, largest value of k is 4

Summary of Arithmetic, Geometric & Quadratic (k=0)

	Arithmetic	Geometric	Quadratic
How to identify	2 4 6 8 10 … +2 +2 +2 +2 Difference constant	1 2 4 8 16 … ×2 ×2 ×2 ×2 Constant ratio	1 3 7 13 21 2 4 6 8 +2 +2 +2 constant second difference
Expression of Single term	$a_0 + dk$	$a_0 \cdot r^k$	$ak^2 + bk + c$
Computing Partial Sum	$\sum_{k=0}^{N} a_0 + dk = (a_0 + a_N)\left(\frac{N+1}{2}\right)$	$\sum_{k=0}^{N} a_0 r^k = \frac{a_0(1-r^{N+1})}{1-r}$	Calculus fun (not CS1800)

Exercise

1) $\sum_{k=0}^{100} 4-k$

$a_0 = 4 \qquad a_{100} = -96$

$(4+-96)\left(\frac{100+1}{2}\right) = \boxed{-4646}$

2) $\sum_{k=0}^{10} 10\cdot 3^k$

$a_0 = 10 \qquad r = 3$

$\frac{10(1-3^{10+1})}{1-3} = \boxed{885730}$

3) $10+7+4+1+(-2)+(-5)+(-8)$

highest → 6

$\sum_{k=0}^{6} 10-3k$

$a_0 = 10 \qquad a_6 = -8$

$(10+-8)\left(\frac{6+1}{2}\right) = \boxed{7}$

Summary of Arithmetic, Geometric & Quadratic

$(k=1)$

	Arithmetic	Geometric	Quadratic
How to identify	2 4 6 8 10 ... +2 +2 +2 +2 Difference constant	1 2 4 8 16 ... ×2 ×2 ×2 ×2 Constant ratio	1 3 7 13 21 2 4 6 8 +2 +2 +2 constant second difference
Expression of Single term	$a_0 + d(k-1)$	$a_0 \cdot r^{k-1}$	$ak^2 + bk + c$
Computing Partial Sum	$\sum_{k=1}^{N} a_0 + dk = (a_0 + a_N)\left(\frac{N}{2}\right)$	$\sum_{k=1}^{N} a_0 r^k = \frac{a_0(1-r^N)}{1-r}$	Calculus fun (not CS1800)