CS1800 Day 17

Admin:
- HW6 due today
- HW7 (induction) released today (due next Friday)
    - slightly shorter than most:
        - more time to prep for exam2
        - will only count as 80% of other HWs with 100 points (HW7 has only 80 points)
- exam2 is next Friday in class
    - practice exam2 problems (and solution) available now
        - prep tip: don't peek at solutions before you've given a problem your best effort

Content:
- Summation Notation
- Strong induction
- Induction: inequality

Exam2: outline

- ☐ one induction problem (equality or inequality)
- ☐ BFS / DFS orderings
- ☐ Dijkstra's Shortest Path Problem (show all steps, as shown in HW)
- ☐ Bayes Rule Problem
- ☐ Expected Value / Variance Problem
- ☐ Counting style probability (each outcome equally likely)

SUMMATION NOTATION

$$1+2+4+8+16+32+64$$

$$\begin{array}{ccccccc} k=0 & k=1 & k=2 & k=3 & & & k=6 \\ = 2^0 + & 2^1 + & 2^2 + & 2^3 + & 2^4 + & 2^5 + & 2^6 \end{array}$$

LAST VALUE OF K

$$= \sum_{k=0}^{6} 2^k$$

BY PUTTING A PARTICULAR K INTO THIS TEMPLATE, YOU CAN PRODUCE ONE TERM

STARTING VALUE OF K

"The sum of 2^k where k goes from 0 to 6"

$$1 + 4 + 7 + 10 + 13 + 16$$

$$\sum_{k=0}^{5} 3k+1$$

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
1	4	7	10	13	16

In Class Activity: Summation Notation

$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$9 +$	$11 +$	$13 +$	$15 +$	17

Express each sum below in summation notation

$$9+11+13+15+17$$

$$\sum_{k=0}^{4} 2k+9$$

$$1+2+3+4+5+\dots+n$$

$$\sum_{k=0}^{n-1} k+1 = \sum_{k=1}^{n} k$$

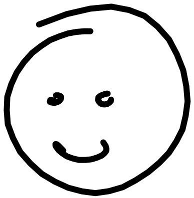
Compute each sum below (the second one has a pattern and simplifies)

$$\sum_{k=10}^{12} 2k = 2\cdot 10 + 2\cdot 11 + 2\cdot 12$$

$$= 2\cdot 3\cdot 11 = 66$$

$$\sum_{k=0}^{101} (-1)^k = 0$$

$$\sum_{k=0}^{101} (-1)^k = 0$$



0 1
2 3
4 5
⋮
98 99
100 101

$k=0$	$k=1$	$k=2$	$k=3$
$(-1)^0$	$(-1)^1$	$(-1)^2$	$(-1)^3$
1	-1	1	-1

$$1 \xrightarrow{\times -1} -1 \xrightarrow{\times -1} 1 \xrightarrow{\times -1} -1$$

Common Summation Notation Manipulation in Induction Proofs: trimming the last term

The black writing below if excerpt from last class's final ICA. The blue text says the same using summations.

Assume $1+2+3+4+\ldots+n = \dfrac{n(n+1)}{2}$

$$\sum_{k=1}^{n} k = n(n+1)/2$$

Then $1+2+3+4+\ldots+n+(n+1) = \dfrac{n(n+1)}{2}+(n+1)$

$$\sum_{k=1}^{n+1} k = \left(\sum_{k=1}^{n} k\right) + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

You can trim off last term (immediately above this text) from a summation notation.

Often helpful to apply inductive hypothesis (assumption)

Examining different induction structures: making change with 3 and 4 cent pieces

Claim: Using only 3 and 4 cent coins, one can produce any whole-number of cents greater than or equal to 6

Proof:

Statement n: there exists a way to produce exactly n cents using 3 and 4 cent coins

Base Cases (there are many):

6 cents = 3 + 3   7 cents = 3 + 4   8 cents = 4 + 4

Induction Step: If statement 6, 7, 8, 9, ..., n are all true, then statement n + 1 is true

Assume: some combo of 3 and 4 cent coins produce 6 cents, 7 cents, 8 cents, ... , n cents

Case 1: the combo of 3 and 4 cents to produce n cents includes a 3 cent coin

- replace this 3 cent coin with a 4 cent coin: new combo produces n + 1 cents

Case 2: the combo of 3 and 4 cents to produce n cents doesn't include a 3 cent coin

- it must contain at least two 4 cent coins (n is at least 8, see base cases above)
- replace these two 4 cent coins with three 3 cent coins: new combo produces n + 1 cents

Induction (Strong):

Induction allows us to prove a never-ending sequence of statements: S(1), S(2), S(3), S(4), ...

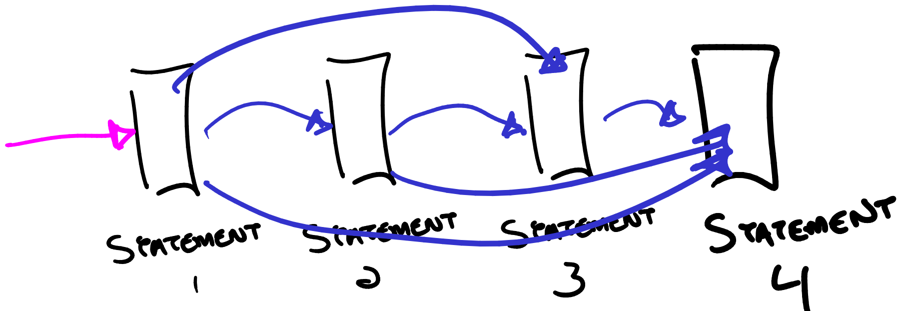
Process:

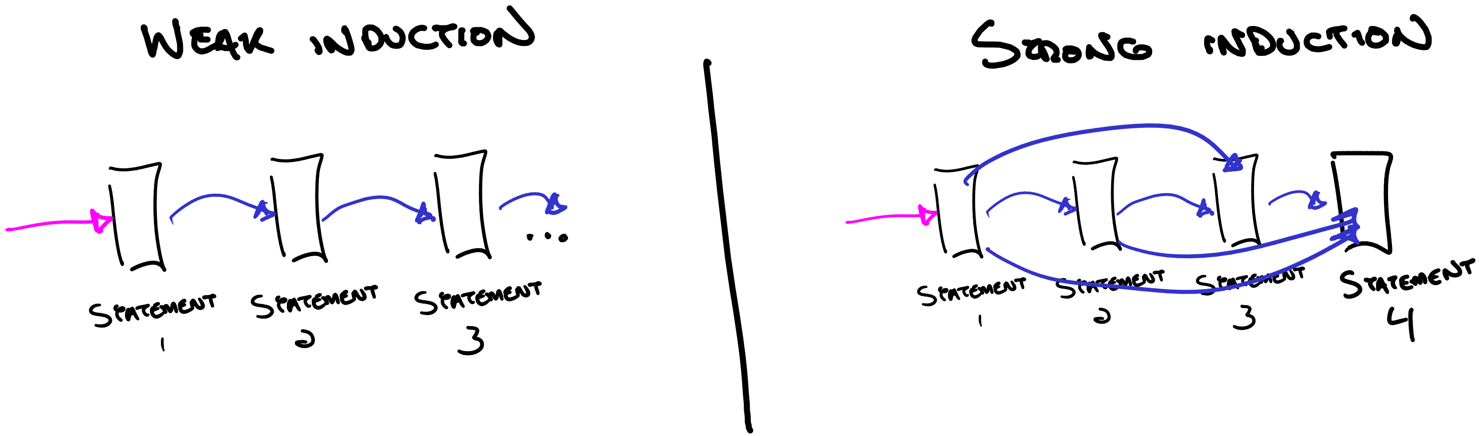
- Prove the first statement, S(n) for some n
- Show that S(1), S(2), ... S(n) implies S(n+1)

Metaphor (Dominos):

To knock over all the dominos

- Push over the first one
- Place each other domino so that if ALL dominos behind it falls, it too will fall





When should I use weak vs strong induction?

Both are always available to you, you may find one method produces a simpler proof (usually weak induction, if it can get the job done).

In Class Acitivy: Why do we study Induction with inequalities?

Suppose two algorithms both accomplish the same task but take a different number of "computes" to do so.

For an input of size N:

- Algorithm 1 takes: 2^N computes 2^N
- Algorithm 2 takes: N! computes $N!$

Input Size	1	2	3	4	5	6
Algorithm 1 Compute Operations	2	4	8	16	32	64
Algorithm 2 Compute Operations	1	2	6	24	120	720

- Complete the table above
- which algorithm would you prefer for a list of size n=2?

algorithm 2 uses n! = 2 operations while algorithm 1 uses 2^N = 4 operations, algorithm 2 is faster

- which algorithm would you prefer for a list of size n=5?

algorithm 1 uses 2^n = 32 operations while algorithm 2 uses N! = 120, algorithm 1 is faster

- if you had to pick one algorithm for lists of any size, which would you choose, why?

n! > 2^n for n which are sufficiently large (algorithm 1, 2^N, will use fewer computations for lists which are large enough)

Induction with inequalities:

$S(n) =$ " $2^N < N!$ "

Prove that 2^N < N! for all N above some threshold.

BASE $N=4$ $\quad 2^N = 2^4 = 16 < 24 = 4! = N!$

INDUCTIVE STEP $\quad S(N) \rightarrow S(N+1)$

ASSUME $2^N < N!$

$2^{N+1} = 2^N \cdot 2$

$< N! \cdot 2$

$< N!(N+1) = (N+1)!$

$2 < N+1$

Induction Recipe (from previous lesson)

1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if S(n) then S(n+1)
4. Prove inductive step:
   a. assume statement n (inductive hypothesis)
   b. write statement n + 1 in two halves
      (tip: start at sum side, work to other side)
   c. apply assumption to get from one half to other

Induction with inequalities: sol

$S(n) = \text{"} 2^N < N! \text{"}$

Prove that 2^N < N! for all N above some threshold.

STATEMENT N: $2^N < N!$

BASE CASE $N = 4$

$2^N = 16 < 24 = N!$

INDUCTIVE STEP $S(N) \to S(N+1)$

ASSUME $2^N < N!$

$$2^{N+1} = 2^N \cdot 2 < N! \cdot 2 < N! \cdot (N+1) = (N+1)!$$

Induction Recipe (from previous lesson)

1. define & remind: statement n
2. choose base case n & show it
3. write "inductive step: if S(n) then S(n+1)
4. Prove inductive step:
    a. assume statement n (inductive hypothesis)
    b. write statement n + 1 in two halves
        (tip: start at sum side, work to other side)
    c. apply assumption to get from one half to other

Algebra: Working with inequalities (1 of 4)

Move 1: add the same things to both sides, it preserves the inequality

IF $3 < 4$ THEN $3+10 < 4+10$

$$x < y \Rightarrow x+c < y+c \quad \forall c \in \mathbb{R}$$

Algebra: Working with inequalities (2 of 4)

Move 2: multiply by a positive value, it preserves the inequality

IF $3 < 4$ THEN $3 \cdot 10 < 4 \cdot 10$

IF $x < y$ THEN $xc < yc$ $\forall c \in \mathbb{R}$ WITH $c > 0$

Move 3: multiply by a negative value, it swaps the inequality direction

IF $3 < 4$ THEN $3 \cdot -1 > 4 \cdot -1$

IF $x < y$ THEN $xc > yc$ $\forall x, y \in \mathbb{R}$ WITH $c < 0$

Algebra: Working with inequalities (3 of 4)

Move 4: sum two inequalities (large side together & small side together)

IF $3 < 4$ AND $5 < 6$ THEN $3+5 < 4+6$

IF $X < Y$ AND $W < Z$ THEN $X+W < Y+Z$

Algebra: Working with inequalities (4 of 4)

Move 5 (another view of move 4 really):

- you can replace a term in smaller side of inequality with something smaller

$$3 < 7 \quad \text{AND} \quad 1 < 3 \quad \Rightarrow \quad 1 < 7$$

$$y < z \quad \text{AND} \quad x < y \quad \Rightarrow \quad x < z$$

- you can also replace a term in larger side of inequality with something larger

$$\text{BABY} < \text{SAL} \quad \text{AND} \quad \text{SAL} < \text{ELI} \quad \Rightarrow \quad \text{BABY} < \text{ELI}$$

Tip: This is one of the most common manipulations in inequality induction problems

In Class Activity:

Complete the table:

Input Size	1	2	3	4	5	6
N^2	~~2~~ 1	4	9	16	25	36
N + 10	11	12	13	14	15	16

Show that N^2 > N + 10 for all N above some value

$S(N)$: $N^2 > N+10$

BASE CASE $N=4$

$N^2 = 4^2 = 16 > 14 = 4+10 = N+10$

INDUCTIVE STEP $S(N) \rightarrow S(N+1)$

ASSUME $N^2 > N+10$

$(N+1)^2 = N^2 + 2N + 1 > N+10 + 2N + 1 = N + 11 + 2N > N + 11$

BIG

SMALL

In Class Activity: sol

Complete the table:

Input Size	1	2	3	4	5	6
N^2	2	4	9	16	25	36
N + 10	11	12	13	14	15	16

Show that N^2 > N + 10 for all N above some value

STATEMENT N: $N+10 < N^2$

BASE CASE $N=4$

$N+10 = 14 < 16 = N^2$

INDUCTIVE STEP: $S(N) \rightarrow S(N+1)$

ASSUME $N+10 < N^2$

$(N+1)+10 < N^2+1$

$< N^2+2N+1$

$= (N+1)^2$

WE ADD A POSITIVE VALUE ($2N$) TO LARGER SIDE (MOVE 5)