Exam2 (Practice)

Due: do-not-submit

Exam Instructions:

- **Please write your name & NUID in the box on top of this page**
- This exam is a strictly individual effort, you may not view or share exam content (including discussing exam questions with other students) until grades have been released.
- Unless otherwise specified, be sure to show your detailed work to earn full credit
- You are welcome to use a calculator without any communication abilities (e.g. no cellphones, or wifi / bluetooth enabled calculators).
- To keep the exam fair to all students, no math-content related help will be offered to students. We are, however, happy to clarify any instructions but please don't expect any hints.
- Your work must be easiliy legible to be graded.
- When the final exam time has been called, please promptly turn in your work. Exams which are submitted late may be penalized 10 points per minute late (in practice we are not likely to apply this penalty so long as you stop working and get up to hand in your exam when time is called).

- $E[x] = \sum_x x * P(X = x)$
- $\texttt{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2$

	No Repeat Selections	Repeat Selections
Order Matters	Permutations: $P(n,k) = \frac{n!}{(n-k)!}$	Product Rule: n^k
Order Doesn't Matter	Combinations: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$	Stars and Bars (Partitions of k identical objects into n groups): $\binom{k+n-1}{n-1}$

Note

- The instructions above are identical to exam2's, they're included for your reference.
- Many of these problems are redundant with each other, but please don't be intimidated by having so many problems here on the practice exam, the real exam is certainly shorter! If you feel confident with the first problem of some type, feel free to skip the second.
- We have an equality and an inequality style induction problem here. On the exam you'll only have one or the other.

Happy studying :)

Problem 1 Counting Cards

Given a standard 52-card deck of playing cards (half of cards are red and half are black) one draws 5 cards without replacement. What is the probability that the first three cards drawn are red, and the last two cards drawn are black? (Notice: this event does not include drawing two black cards before then three red cards).

Be sure to show and justify and clearly document your work as well as computing a final probability value.

Solution

There are $P(52,5)$ orderings of 5 cards from the deck, each is equally likely as all the others.
There are $P(26,3)$ ways of ordering 3 red cards and $P(26,2)$ ways of ordering 2 black cards. Therefore, there are $P(26,3) * P(26,2)$ ways of choosing 3 red cards and then 2 black cards.

The probability of our event is then:

$$P(x=1)_{=}\frac{P(26,3)*P(26,2)}{P(52,5)}\approx .0325$$

Some students may have an alternate solution:

$$P(x=1)=\frac{26}{52}*\frac{25}{51}*\frac{24}{50}*\frac{26}{49}*\frac{25}{48}$$

The first three fractions are the probabilities of pulling 3 red cards and the last two fractions are the probabilities of pulling 2 black cards. Notice that the solutions are equivilent since:

- $P(26,3)=26*25*24$
- $P(26,2)=26*25$
- $P(52,5)=52*51*50*49*48$

Problem 2 Expected Value & Variance: Call Center Wait Times

An internet service provider's call center receives calls which are each one of the following types:

Call Type	Time of Call (min)	Probability
General Inquiry	2	0.20
Technical Support	4	0.25
Billing Questions	3	0.15
Service Activation	7	0.10
Device Troubleshooting	5	0.30

Table 1: Probability of Response Time for Call Center (Sum to 1)

i Compute the expected time it takes to handle a single call

Solution

$$E[x]=\sum_{x}x*P(x)=2*.2+4*.25+3*.15+7*.1+5*.3=4.05$$

ii Compute the variance in time it takes to handle a single call

Solution

$$E[x^2] = 2^2 * .2 + 4^2 * .25 + 3^2 * .15 + 7^2 * .1 + 5^2 * .3 = 18.55$$

so that:

$$Var(x) = E[x^2] - E[x]^2 = 18.55 - 4.05^2 = 2.1475$$

iii An employee working at the center is able to handle 15 calls in an hour (this employee's calls follow the same distribution as above). Is this employee typically faster or slower than others in the group? Explain

Solution

The average employee takes 4.05 minutes per call, handling $60/4.05 \approx 14.8$ calls per hour. This particular employee is faster.

Problem 3 Expected Value & Variance: Basketball

A basketball player makes shots according to the following table:

	Distance to net	**Points earned (if made)**	**Prob shot made (%)**
P	In Paint	2	45
J	Mid Range Jump	2	40
T	3 pt	3	35

Where we use random variables P, J or T to indicate the number of points earned when a player takes each of the corresponding shots.

i Compute the expected value and variance of: P

Solution

$$E[P] = .45 * 2 + .55 * 0 = .9$$
$$\texttt{Var}(P) = .45(2 - .9)^2 + .55 * (0 - .9)^2 = .99$$

ii Compute the expected value and variance of: J

Solution

$$E[J] = .4 * 2 + .6 * 0 = .8$$
$$\texttt{Var}(J) = .4(2 - .8)^2 + .6 * (0 - .8)^2 = .96$$

iii Compute the expected value and variance of: T

Solution

$$E[T] = .35 * 3 + .65 * 0 = 1.05$$
$$\texttt{Var}(T) = .35(3 - 1.05)^2 + .65 * (0 - 1.05)^2 = 2.0475$$

iv If your team was down by 1 point with time for 1 more shot, which shot should this player prefer to maximize their chance of winning?

Solution

Any made shot wins the game. This player is most likely to hit a shot in the paint: P.

v In the early game, which shot should this player prefer?

Solution

In the early game, we're interested in scoring as much as possible. This player has the highest expected points earned with their three point shot.

Problem 4 When it rains, park wherever you'd like

A license plate reader is a system which identifies the text of a license plate given a camera's image of the back of a car (useful in parking enforcement). When its not raining, the system is able to identify 99% of the license plates correctly. Unfortunately, when it rains the system struggles to see clearly and only identifies 90% of the license plates correctly. In a particularly rainy parking lot, there is a $\frac{1}{4}$ chance of rain each day.

i What is the probability that its raining and a license isn't read correctly?

Solution

$$P(R = 1, C = 0) = P(C = 0|R = 1)P(R = 1) = .1 * .25 = .025$$

ii What is the probability that its not raining and a license isn't read correctly?

Solution

$$P(R = 0, C = 0) = P(C = 0|R = 0)P(R = 0) = .01 * .75 = .0075$$

iii What is the probability that a license isn't read correctly?

Solution

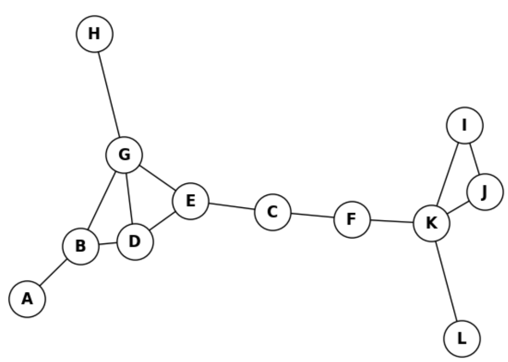
$$P(C = 0) = P(C = 0, R = 0) + P(C = 0, R = 1) = .025 + .0075 = .0325$$

iv Suppose that you receive a parking ticket for someone else's car because their license plate was incorrectly read as yours. What is the probability that it was raining when this other car's license plate was read incorrectly?

Solution

$$\begin{aligned} P(R = 1|C = 0) &= \frac{P(C = 0|R = 1)P(R = 1)}{P(C = 0)} \\ &= \frac{(.1)(.25)}{.0325} \\ &\approx .77 \end{aligned}$$

Problem 5 Breadth / Depth First Search



Wherever possible below, select the node which is earlier in the alphabet first (e.g. prefer visiting node A first over node B, when the search allows you to visit either).

i Starting at L, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

LKFI JCED GBHA

ii Starting at A, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

ABDG EHCF KIJL

iii Starting at D, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

DBEG ACHF KIJL

iv Starting at L, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

LKFC EDBA GHIJ

v Starting at A, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

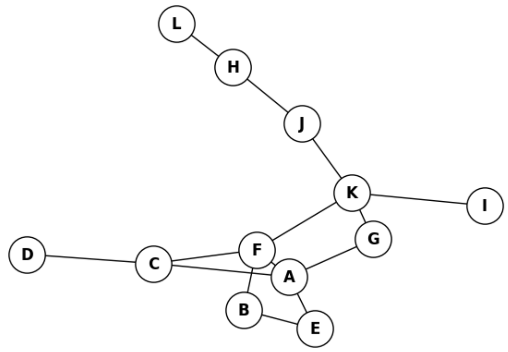
ABDE CFKI JLGH

vi Starting at D, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

DBAG ECFK IJLH

Problem 6 Breadth / Depth First Search



Wherever possible below, select the node which is earlier in the alphabet first (e.g. prefer visiting node A first over node B, when the search allows you to visit either).

i Starting at L, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

LHJK FGIA BCED

ii Starting at D, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

DCAF EGBK IJHL

iii Starting at E, find the Breadth-First-Search (BFS) ordering of nodes in the graph above.

Solution

EABC FGDK IJHL

iv Starting at L, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

LHJK FACD EBGI

v Starting at D, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

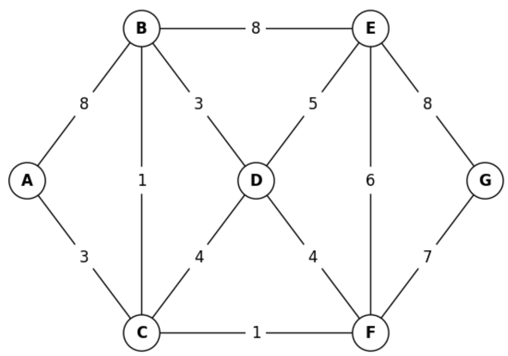
DCAE BFKG IJHL

vi Starting at E, find the Depth-First-Search (DFS) ordering of nodes in the graph above.

Solution

EACD FBKG IJHL

Problem 7 Dijkstra's Shortest Path



Using Dijkstra's algorithm, find the shortest path from node A to G. Please provide a table which shows the path weight and predecessor from A to every node, labelling the visited node at each step. an example solution is given here.

Solution

iteration	node visited	A	B	C	D	E	F	G
0	A	start:0	A: 8	A: 3	none	none	none	none
1	C	start:0	C: 4	A: 3	C: 7	none	C: 4	none
2	B	start:0	C: 4	A: 3	C: 7	B: 12	C: 4	none
3	F	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11
4	D	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11
5	E	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11

The path with min weight is: G $\leftarrow$ F $\leftarrow$ C $\leftarrow$ A

The solution immediately above is sufficient for full credit.A step-by-step explanation is given below:

- Start algorithm: visit start node A
- add path A → B with cost 8 (no previous path to B)
- add path A → C with cost 3 (no previous path to C)

iteration	node visited	A	B	C	D	E	F	G
0	A	start:0	A: 8	A: 3	none	none	none	none

- Continue algorithm: visit node with min cost among unvisited: C has cost 3
- update to path A → C → B with cost 4 (previous path A → B had cost 8)
- add path A → C → D with cost 7 (no previous path to D)
- add path A → C → F with cost 4 (no previous path to F)

iteration	node visited	A	B	C	D	E	F	G
1	C	start:0	C: 4	A: 3	C: 7	none	C: 4	none

- Continue algorithm: visit node with min cost among unvisited: B has cost 4
- ignore path A → B → D with cost 7 (previous path A → C → D had cost 7)
- add path A → B → E with cost 12 (no previous path to E)

iteration	node visited	A	B	C	D	E	F	G
2	B	start:0	C: 4	A: 3	C: 7	B: 12	C: 4	none

- Continue algorithm: visit node with min cost among unvisited: F has cost 4
- ignore path A → ... → F → D with cost 8 (previous path A → C → D had cost 7)
- update to path A → ... → F → E with cost 10 (previous path A → B → E had cost 12)
- add path A → ... → F → G with cost 11 (no previous path to G)

iteration	node visited	A	B	C	D	E	F	G
3	F	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11

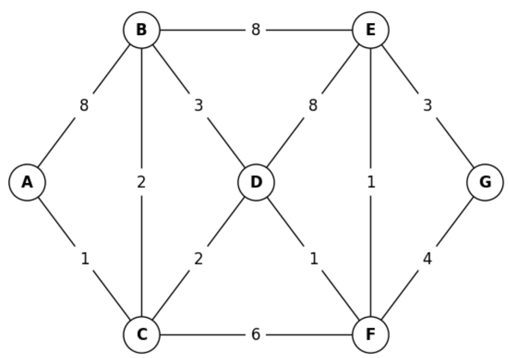
- Continue algorithm: visit node with min cost among unvisited: D has cost 7
- ignore path A → ... → D → E with cost 12 (previous path A → ... → F → E had cost 10)

iteration	node visited	A	B	C	D	E	F	G
4	D	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11

- Continue algorithm: visit node with min cost among unvisited: E has cost 10
- ignore path A → ... → E → G with cost 18 (previous path A → ... → F → G had cost 11)
- End algorithm: next to visit is destination: G has cost 11

iteration	node visited	A	B	C	D	E	F	G
5	E	start:0	C: 4	A: 3	C: 7	F: 10	C: 4	F: 11

Problem 8 Dijkstra's Shortest Path



Using Dijkstra's algorithm, find the shortest path from node A to G. Please provide a table which shows the path weight and predecessor from A to every node, labelling the visited node at each step. an example solution is given here.

Solution

iteration	node visited	A	B	C	D	E	F	G
0	A	start:0	A: 8	A: 1	none	none	none	none
1	C	start:0	C: 3	A: 1	C: 3	none	C: 7	none
2	B	start:0	C: 3	A: 1	C: 3	B: 11	C: 7	none
3	D	start:0	C: 3	A: 1	C: 3	B: 11	D: 4	none
4	F	start:0	C: 3	A: 1	C: 3	F: 5	D: 4	F: 8
5	E	start:0	C: 3	A: 1	C: 3	F: 5	D: 4	F: 8

The path with min weight is: G ← F ← D ← C ← A

The solution immediately above is sufficient for full credit.A step-by-step explanation is given below:

- Start algorithm: visit start node A
- add path A → B with cost 8 (no previous path to B)
- add path A → C with cost 1 (no previous path to C)

iteration	node visited	A	B	C	D	E	F	G
0	A	start:0	A: 8	A: 1	none	none	none	none

- Continue algorithm: visit node with min cost among unvisited: C has cost 1
- update to path A → C → B with cost 3 (previous path A → B had cost 8)
- add path A → C → D with cost 3 (no previous path to D)
- add path A → C → F with cost 7 (no previous path to F)

iteration	node visited	A	B	C	D	E	F	G
1	C	start:0	C: 3	A: 1	C: 3	none	C: 7	none

- Continue algorithm: visit node with min cost among unvisited: B has cost 3
- ignore path A → B → D with cost 6 (previous path A → C → D had cost 3)
- add path A → B → E with cost 11 (no previous path to E)

iteration	node visited	A	B	C	D	E	F	G
2	B	start:0	C: 3	A: 1	C: 3	B: 11	C: 7	none

- Continue algorithm: visit node with min cost among unvisited: D has cost 3
- ignore path A → ... → D → E with cost 11 (previous path A → B → E had cost 11)
- update to path A → ... → D → F with cost 4 (previous path A → C → F had cost 7)

iteration	node visited	A	B	C	D	E	F	G
3	D	start:0	C: 3	A: 1	C: 3	B: 11	D: 4	none

- Continue algorithm: visit node with min cost among unvisited: F has cost 4
- update to path A → ... → F → E with cost 5 (previous path A → B → E had cost 11)
- add path A → ... → F → G with cost 8 (no previous path to G)

iteration	node visited	A	B	C	D	E	F	G
4	F	start:0	C: 3	A: 1	C: 3	F: 5	D: 4	F: 8

- Continue algorithm: visit node with min cost among unvisited: E has cost 5
- ignore path A → ... → E → G with cost 8 (previous path A → ... → F → G had cost 8)
- End algorithm: next to visit is destination: G has cost 8

iteration	node visited	A	B	C	D	E	F	G
5	E	start:0	C: 3	A: 1	C: 3	F: 5	D: 4	F: 8

Problem 9 Induction (equality): $\sum k^2$

The sum of the squares of the first n positive integers may be computed directly

$$1^2 + 2^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Prove the equality above using induction.

Solution

"Reminder" Statement n:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Base Step: Statement 1

If $n = 1$ then:

$$\sum_{k=1}^{1} k^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

Inductive Step: (if statement n then statement $n+1$)

For some $n \in \mathbb{N}$, we assume statement n holds:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

We compute:

$$\begin{aligned}
\sum_{k=1}^{n+1} k^2 &= (n+1)^2 + \sum_{k=1}^{n} k^2 \\
&= (n+1)^2 + \frac{n(n+1)(2n+1)}{6} \\
&= \frac{6(n+1)^2 + n(n+1)(2n+1)}{6} \\
&= \frac{(n+1)(6(n+1) + n(2n+1))}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

Problem 10 Induction (inequality): $n! < n^n$
Using induction, show that $n! < n^n$ for all $n = 2, 3, 4, \ldots$.

Solution

"Reminder" S(n):

$$n! < n^n$$

Base Step P(2):

$$2! = 2 * 1 = 2 < 4 = 2^2$$

so that $2! < 2^2$

Inductive Step $S(n) \rightarrow S(n+1)$**:**

We assume $S(n)$, that $n! < n^n$ for some n. Then we may compute:

$$\begin{aligned}(n+1)! &= n! * (n+1) \\ &< n^n * (n+1) \\ &< (n+1)^n * (n+1) \\ &= (n+1)^{n+1}\end{aligned}$$

Where we use $S(n)$ in the second line and the fact that $n^n < (n+1)^n$ in the third (since n is positive).
By induction, $n! < n^n$ for all $n > 1$.