CS1800 Day 18

Admin:

- hw7 (induction) due Friday
- exam2 on Friday
    - this material (day 18) is not on exam2
- recitation this week:
    - no quiz
    - focus on exam2 practice problems (available on website)

Content:

- Series & Sequences (Arithmetic, Geometric & Quadratic)
- Given a series, identify its type (may be none of the 3 above)
- Express the i-th term in a sequence
- Compute the partial sum of a series (Arithmetic & Geometric)

Summation Notation: a quick reminder

K IN LAST TERM

$$\sum_{K=0}^{4} 1+2^K = \begin{aligned} &1+2^0 \\ +\ &1+2^1 \\ +\ &1+2^2 \\ +\ &1+2^3 \\ +\ &1+2^4 \end{aligned} = \begin{aligned} &1 \\ +\ &3 \\ +\ &5 \\ +\ &9 \\ +\ &17 \end{aligned} = 35$$

K IN FIRST TERM

NOTICE: K IS WHOLE NUMBER WHICH STEPS BY 1

Sequences & Series (definition):

A sequence is an ordered list of objects (always numbers in this CS1800 unit)

$$1, 2, 3, 4, 5, 6, \ldots$$

A series is the sum of an infinite sequence of objects

$$1+2+3+4+5+6+\ldots = \sum_{k=1}^{\infty} k$$

A partial sum (of a series) is the sum of part of a series

$$1+2+3+4 = \sum_{k=1}^{4} k = 10$$

Arithmetic Sequence / Series: What it is (and how to identify it)

An arithmetic sequence's first difference (next term - current term) is constant:

$$10 \xrightarrow{+2} 12 \xrightarrow{+2} 14 \xrightarrow{+2} 16 \xrightarrow{+2} 18 \xrightarrow{+2} 20 \quad \ldots$$

FIRST DIFFERE

To test if a sequence is arithmetic, compute first difference. If its constant then sequence is arithmetic.

$$11 \xrightarrow{-7} 4 \xrightarrow{-7} -3 \xrightarrow{-7} -10 \xrightarrow{-7} -17$$

Arithmetic Series / Partial Sum: What do they look like in summation notation?

Example:

$$10 \overset{+2}{\rightarrow} 12 \overset{+2}{\rightarrow} 14 \overset{+2}{\rightarrow} 16 + \ldots = \sum_{k=0}^{\infty} 10 + 2k$$

$10+2\cdot 0$ $\quad$ $10+2\cdot 1$ $\quad$ $10+2\cdot 2$ $\quad$ $10+2\cdot 3$

$$a_k = 10 + 2k$$

$$a_0 \quad a_1 \quad a_2 \quad a_3$$

Every arithmetic series can be expressed via the following form:

STARTING VALUE

INDEX

$$\sum_{k=0}^{\infty} a_0 + dk$$

DIFFERENCE BETWEEN ADJACENT VALUES

Geometric Sequences / Series: What it is (and how to identify it)

An Geometric sequence is one whose first ratio (next term / current term) is constant:

$$\tfrac{1}{2} \xrightarrow{\times 2} 1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \quad \ldots$$

CONSTANT RATIO

To test if a sequence is geometric, compute first ratio. If its constant then sequence is geometric.

$$100 \xrightarrow{\times -\frac{1}{10}} -10 \xrightarrow{\times -\frac{1}{10}} 1 \xrightarrow{\times -\frac{1}{10}} -\frac{1}{10} \quad \frac{1}{100} \quad \ldots$$

Geometric Series / Partial Sum: What do they look like in summation notation?

Example:

$k=0 \quad k=1 \quad k=2 \quad k=3$

$\times 2 \quad \times 2 \quad \times 2$

$$\frac{1}{2} + 1 + 2 + 4 + 8 + \dots = \sum_{k=0}^{\infty} \frac{1}{2} \cdot 2^k$$

$\frac{1}{2}\cdot 2^0 \qquad \frac{1}{2}\cdot 2^1 \qquad \frac{1}{2}\cdot 2^2 \qquad \frac{1}{2}\cdot 2^3 \qquad \frac{1}{2}\cdot 2^4$

$$a_k = a_0 \cdot r^k$$

Every geometric series can be expressed via the following form:

$$\sum_{k=0}^{\infty} a_0 \cdot r^k$$

INDEX

RATIO OF $\dfrac{\text{NEXT TERM}}{\text{CURRENT TERM}}$

STARTING TERM

Quadratic Series / Partial Sum: What is it? (i.e. what does it look like in sum notation?)

Every quadratic series can be expressed as:

$$\sum_{k=0}^{\infty} a\underline{k^2} + b\underline{k} + c$$

FIRST TERM

a, b, c ARE CONSTANTS

(NOT A EASILY SEEN AS ARITHMETIC / GEOMETRIC)

Example (a=1, b=0, c=0):

$1\cdot 2^2 + 0\cdot 2 + 0$ $\quad$ $1\cdot 4^2 + 0\cdot 4 + 0$

$$0 + 1 + 4 + 9 + 16 + 25 + \ldots$$

$\underline{1}\cdot \underline{0^2} + \underline{0}\cdot \underline{0} + \underline{0}$ $\quad$ $1\cdot 1^2 + 0\cdot 1 + 0$ $\quad$ $1\cdot 3^2 + 0\cdot 3 + 0$ $\quad$ $1\cdot 5^2 + 0\cdot 5 + 0$

Question (for later): given the first few values in sequence, how can we get a, b, c?

Quadratic Sequences / Series: How to identify it

The second difference of a quadratic sequence is constant

1 3 7 13 21 31 . . .

First Difference

+2 +4 +6 +8 +10

Second Difference

+2 +2 +2 +2

In Class Activity:

Identify the type (arithmetic, geometric, quadratic, or none) of each of the following sequences. If sequence is arithmetic or geometric, express its corresponding series in sum notation.

QUADRATIC

i. 6, 15, 28, 45, 66, 91, ...

+9 +13 +17 +21 +25

+4 +4 +4 +4

ii. 1, -4, 16, -64, 256, ...

x-4 x-4 x-4

GEOMETRIC

$$\sum_{k=0}^{\infty} 1\cdot(-4)^k$$

iii. 4, 7, 10, 13, 16, 19, ...

+3 +3 +3 +3

CONSTANT 1st DIFF → ARITHMATIC

$$\sum_{k=0}^{\infty} 4+3k$$

iv. 2, 7, 11, 42, -4, ...

NOT ARITHMETIC / GEOMETRIC OR QUADRATIC

In Class Activity:

Identify the type (arithmetic, geometric, quadratic, or none) of each of the following sequences. If sequence is arithmetic or geometric, express its corresponding series in sum notation.

i. 6, 15, 28, 45, 66, 91, … QUADRATIC

+9 +13 +17 +21

+4 +4 +4

ii. 1, -4, 16, -64, 256, … GEOMETRIC $\sum_{k=0}^{\infty} (-4)^k$

x-4 x-4 x-4

iii. 4, 7, 10, 13, 16, 19, … ARITHMETIC $\sum_{k=0}^{\infty} 4+3k$

+3 +3 +3

iv. 2, 7, 11, 42, -4, … NONE

+5 +4 +31

-1 +27

Quadratic Series: Given sequence, how to compute a, b, c in summation notation

$k=0$, $k=1$, $k=2$, $k=3$, $k=4$, $k=5$

$$6 + 15 + 28 + 45 + 66 + 91 + \ldots = \sum_{k=0}^{\infty} ak^2 + bk + c$$

$$6 = a\cdot 0^2 + b\cdot 0 + c \to c = 6$$

$$15 = a\cdot 1^2 + b\cdot 1 + c \to 15 = a+b+6 \to 9 = a+b \to b = 9-a$$

$$28 = a\cdot 2^2 + b\cdot 2 + c \to 28 = 4a+2b+6 \to 22 = 4a+2b$$

$$\to 11 = 2a+b$$

$$11 = 2a+9-a$$

$$2 = a$$

$$9 = 2+b \quad b = 7$$

Quadratic Series: Given sequence, how to compute a, b, c in summation notation

$k=0 \quad k=2 \quad k=4$

$$6 + 15 + 28 + 45 + 66 + 91 + \ldots = \sum_{k=0}^{\infty} ak^2 + bk + c$$

$k=1 \quad k=3 \quad k=5$

$6 = a\cdot 0^2 + b\cdot 0 + c$ $\qquad \boxed{c=6}$

$15 = a\cdot 1^2 + b\cdot 1 + c \rightarrow 15 = a+b+6$ $\qquad 9 = a+b$, $b = 9-a$

$28 = a\cdot 2^2 + b\cdot 2 + c \rightarrow 28 = 4a + 2b + 6$ $\qquad 22 = 4a + 2b$, $11 = 2a + b$, $11 = 2a + 9 - a$, $\boxed{2 = a}$

$9 = a + b$

$-2 + b$

$\boxed{7 = b}$

Checking our work with python

(you needn't ever do the same for CS1800 ... but cute to see that you can using python)

```
matt@matt-yoga-nu:~$ python3
Python 3.10.12 (main, Jun 11 2023, 05:26:28) [GCC 11.4.0] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> a, b, c = 2, 7, 6
>>> [a * k ** 2 + b * k + c for k in range(10)]
[6, 15, 28, 45, 66, 91, 120, 153, 190, 231]
>>>
```

→ SAME AS GIVEN :)

If you're interested in doing the same and don't have python on your computer, check out "google colab" which allows you to run python code in the cloud.

Quadratic Series Convention: start counting at k=0 or k=1?

$$6 + 15 + 28 + 45 + 66 + 91 + \dots$$

$k=0 \quad k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5$

$$6 = a \cdot 0^2 + b \cdot 0 + c$$

$$15 = a \cdot 1^2 + b \cdot 1 + c$$

$$28 = a \cdot 2^2 + b \cdot 2 + c$$

$$6 + 15 + 28 + 45 + 66 + 91 + \dots$$

$k=1 \quad k=2 \quad k=3 \quad k=4 \quad k=5 \quad k=6$

$$6 = a \cdot 1^2 + b \cdot 1 + c$$

$$15 = a \cdot 2^2 + b \cdot 2 + c$$

$$28 = a \cdot 3^2 + b \cdot 3 + c$$

Starting to count at k=0 (left) or k=1 (right) yields different a,b,c. Both are correct in their own contexts.
Prefer starting to count at k=0 (left), its easier: that first equation immediately gives c.

In Class Activity

Find the coefficients (a, b, c) which allow us to express the following series in summation notation (convention: first term has k=0)

$$\sum_{k=0}^{\infty} ak^2 + bk + c$$

1 + 3 + 7 + 13 + 21 + 31 + 43 + 57 + 73 + 91 + …

$1 = a \cdot 0^2 + b \cdot 0 + c \rightarrow c = 1$

$3 = a \cdot 1^2 + b \cdot 1 + c \rightarrow 3 = a + b + 1 \rightarrow b = 2 - a$

$7 = a \cdot 2^2 + b \cdot 2 + c \rightarrow 7 = 4a + 2b + 1 \rightarrow 6 = 4a + 2b$

$3 = 2a + b$

$3 = 2a + 2 - a$

$1 = a$

$b = 2 - a = 2 - 1 = 1$

In Class Activity

Find the coefficients (a, b, c) which allow us to express the following series in summation notation (convention: first term has k=0)

$1 + 3 + 7 + 13 + 21 + 31 + 43 + 57 + 73 + 91 + \ldots = \sum_{k=0}^{\infty} ak^2 + bk + c$

$+2 \quad +4 \quad +6 \quad +8$

$1 = 0^2 \cdot a + 0 \cdot b + c \rightarrow c = 1$

$3 = 1^2 \cdot a + 1 \cdot b + c \rightarrow 3 = a + b + 1 \rightarrow b = 2 - a$

$7 = 2^2 \cdot a + 2 \cdot b + c \rightarrow 7 = 4a + 2b + 1 \rightarrow 6 = 4a + 2(2-a)$

$= 4a + 4 - 2a$

$2 = 2a$

$1 = a$

$b = 2 - a$
$= 2 - 1 = 1$

Up next: computing partial sums (arithmetic & geometric ... not quadratic)

ARITHMETIC

$$0+1+2+3+4 = \sum_{k=0}^{4} k = ?$$

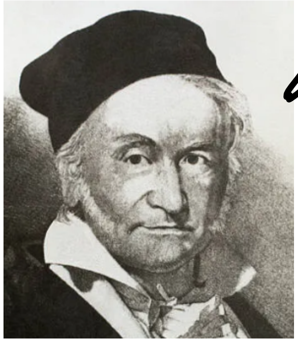
NO SIMPLE FORMULA EXISTS ☹

GEOMETRIC

$$1+2+4+8+16 = \sum_{k=0}^{4} 2^k = ?$$

Computing Arithmetic Partial Series: motivation via tall tale

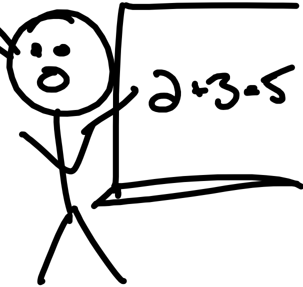
PRIMARY SCHOOL GAUSS



Its 5050

Gauss, you're not paying attention. As punishment go in the hall and add all the integers from 1 to 100

TEACHER



$$0+1+2+\dots+98+99+100$$

$$2+98=100$$

$$1+99=100$$

$$0+100=100$$

50 SUMS OF 100

+ LEFTOVER 50 = 5050

Computing Arithmetic Sums: A more generalizable expression

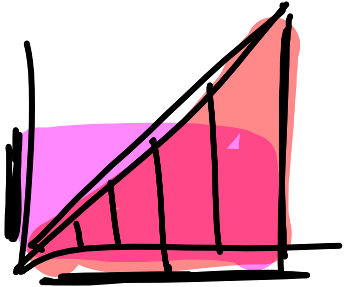
SMALL TEST EXAMPLE

$$1 + 2 + 3 + 4 + 5 = 15$$

AVERAGE TERM $\times$ NUMBER OF TERMS

$$\frac{1+5}{2} = 3 \quad \times \quad 5$$

$$\boxed{\sum_{k=0}^{N} a_0 + dk = \left(\frac{a_0 + a_N}{2}\right) \times (N+1)}$$



Notice:

Summing from k=0 up to N has N + 1 total terms.

(should we choose convention that our first term is k=1, then this formula changes a bit to have N total terms)

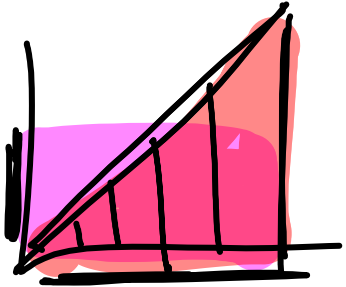
Computing Arithmetic Sums: A more generalizable expression

SMALL TEST EXAMPLE

$$0+1+2+3+4+5 = 15$$

AVERAGE TERM × NUMBER OF TERMS

$$\frac{0+5}{2} = 2.5 \qquad 6$$



$$\boxed{\sum_{k=0}^{N} a_0 + dk = \left(\frac{a_0+a_N}{2}\right) \times (N+1)}$$

Notice:

Summing from k=0 up to N has N + 1 total terms.

(should we choose convention that our first term is k=1, then this formula changes a bit to have N total terms)

Computing Geometric Series Partial Sums

$$S = \sum_{k=0}^{N} a r^k = \underline{a} + ar + ar^2 + \dots + ar^N$$

$$S \cdot r = ar + ar^2 + \dots + ar^N + \underline{ar^{N+1}}$$

S IS THE PARTIAL SUM WE'D LIKE TO COMPUTE

$$S - Sr = a - ar^{N+1}$$

So $S(1-r) = a(1-r^{N+1}) \Rightarrow \boxed{S = \frac{a(1-r^{N+1})}{1-r}}$

Computing Geometric Series: Lets work a little example to check if that formula works

$$\underset{k=0}{1} + \underset{k=1}{2} + \underset{k=2}{4} + \underset{k=3}{8} + \underset{k=4}{16} = \sum_{k=0}^{4} 1 \cdot 2^k = 31$$

$\times 2$ $\times 2$

FIRST TERM

N = LARGEST VALUE OF K IN SUM

$$S = \frac{a_0\left(1 - r^{N+1}\right)}{1-r} = \frac{1 \cdot \left(1-2^5\right)}{1-2} = \frac{-31}{-1} = 31$$

r = RATIO

In summary (Arithmetic, Geometric & Quadratic Sequences / Series / Partial Sums)
(assumes first term is ~~k=0)~~

	Arithmetic	Geometric	Quadratic
How to identify?	2 4 6 8 … +2 +2 +2 CONSTANT FIRST DIFFERENCE	1 2 4 8 16 … ×2 ×2 ×2 ×2 CONSTANT RATIO	1 3 7 13 21 … +2 +4 +6 +8 +2 +2 +2 CONSTANT SECOND DIFFERENCE
Expression of a single term	$a_0 + dK$	$a_0 r^K$	$aK^2 + bK + c$
Computing partial sum	$\sum_{K=0}^{N} a_0 + dK = \underbrace{\left(\frac{a_0 + a_N}{2}\right)}_{\text{AVERAGE TERM}} \cdot \underbrace{(N+1)}_{\text{NUMBER OF TERMS}}$	$\sum_{K=0}^{N} a_0 r^K = \frac{a_0(1 - r^{N+1})}{1 - r}$	KIND OF A CALCULUS THING (NOT NEEDED FOR CS1800)

In summary (Arithmetic, Geometric & Quadratic Sequences / Series / Partial Sums) (assumes first term is k=1)

	Arithmetic	Geometric	Quadratic
How to identify?	$2 \xrightarrow{+2} 4 \xrightarrow{+2} 6 \xrightarrow{+2} 8 \ldots$ CONSTANT FIRST DIFFERENCE	$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \ldots$ CONSTANT RATIO	$1 \xrightarrow{+2} 3 \xrightarrow{+4} 7 \xrightarrow{+6} 13 \xrightarrow{+8} 21 \ldots$ +2 +2 +2 CONSTANT SECOND DIFFERENCE
Expression of a single term	$a_1 + d(k-1)$	$a_1 r^{k-1}$	$ak^2 + bk + c$
Computing partial sum	$\sum_{k=1}^{N} a_1 + d(k-1) = \underbrace{\left(\frac{a_1 + a_N}{2}\right)}_{\text{AVERAGE TERM}} \cdot \underbrace{N}_{\text{NUMBER OF TERMS}}$	$\sum_{k=1}^{N} a_1 r^{k-1} = \frac{a_1(1-r^N)}{1-r}$	KIND OF A CALCULUS THING (NOT NEEDED FOR CS1800)

In Class Activity:

Compute each of the following sums (using the partial sums formula)

i. $\sum_{K=0}^{100} 4-1K = \left(\frac{a_0+a_N}{2}\right)(N+1) = \left(\frac{4+4-100}{2}\right)(101) = -46\cdot 101$

$= -4646$

ii. $\sum_{K=0}^{10} 10\cdot 3^K = \frac{a_0(1-r^{N+1})}{1-r} = \frac{10(1-3^{11})}{1-3}$ = 885730

$K= 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6$

iii. 10 + 7 + 4 + 1 + (-2) + (-5) + (-8)

$\sum_{K=0}^{6} 10-3K = \left(\frac{a_0+a_N}{2}\right)(N+1) = \left(\frac{10-8}{2}\right)(7) = 7$

In Class Activity:

Compute each of the following sums (using the partial sums formula)

i. $\sum_{K=0}^{100} 4 - 1K = \left(\frac{a_0 + a_N}{2}\right) \cdot (N+1) = \frac{4 + (4 - 100)}{2} \cdot 101$

ii. $\sum_{K=0}^{10} 10 \cdot 3^K = \frac{a_0(1 - r^{N+1})}{1 - r} = \frac{10(1 - 3^{11})}{1 - 3}$

iii. 10 + 7 + 4 + 1 + (-2) + (-5) + (-8)

$$\sum_{K=0}^{6} 10 - 3K = \left(\frac{a_0 + a_N}{2}\right) \cdot (N+1) = \frac{10 - 8}{2} \cdot 7$$