

CS1800 Day 18

Admin:

- hw7 (induction) due Friday
- exam2 on Friday
 - this material (day 18) is not on exam2
- recitation this week:
 - no quiz
 - focus on exam2 practice problems (available on website)

Content:

- Series & Sequences (Arithmetic, Geometric & Quadratic)
- Given a series, identify its type (may be none of the 3 above)
- Express the i -th term in a sequence
- Compute the partial sum of a series (Arithmetic & Geometric)

Summation Notation: a quick reminder

A diagram illustrating summation notation. A large grey cloud contains the expression $1 + 2^k =$. A pink arrow points from the text "K IN LAST TERM" to a pink circle containing the number 4. A green arrow points from the text "K IN FIRST TERM" to a green circle containing the expression $k=0$. A wavy line connects the 4 and the $k=0$ to the summation symbol.

$$\sum_{k=0}^4 1 + 2^k =$$

An expansion of the summation notation. The first part shows the terms $1 + 2^0$, $1 + 2^1$, $1 + 2^2$, $1 + 2^3$, and $1 + 2^4$ stacked vertically and separated by plus signs. Blue arrows point from the 2 in each term to the next term below it. This is followed by an equals sign and the terms 1, 3, 5, 9, and 17 stacked vertically and separated by plus signs. A final equals sign is followed by the number 35.

$$\begin{aligned} &+ 1 + 2^0 \\ &+ 1 + 2^1 \\ &+ 1 + 2^2 \\ &+ 1 + 2^3 \\ &+ 1 + 2^4 \end{aligned} = \begin{aligned} &+ 1 \\ &+ 3 \\ &+ 5 \\ &+ 9 \\ &+ 17 \end{aligned} = 35$$

NOTICE: k IS WHOLE NUMBER WHICH STEPS BY 1

Sequences & Series (definition):

A **sequence** is an ordered list of objects (always numbers in this CS1800 unit)

$$1, 2, 3, 4, 5, 6, \dots$$

A **series** is the **sum** of an **infinite** sequence of objects

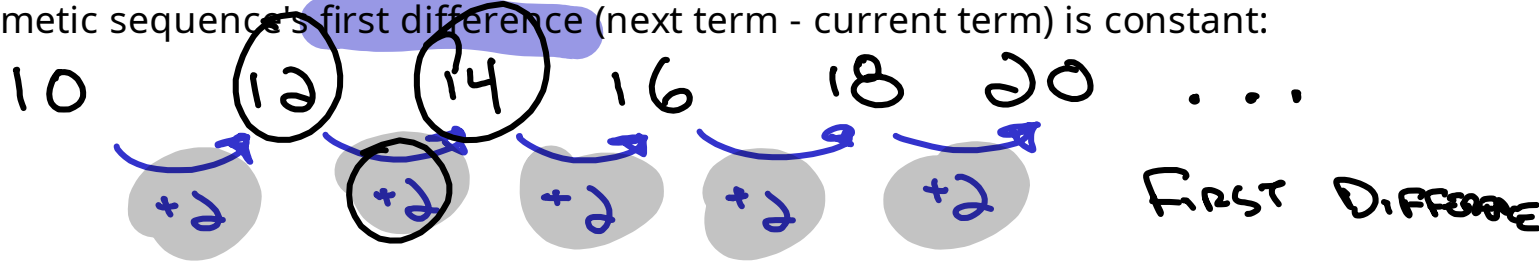
$$1 + 2 + 3 + 4 + 5 + 6 + \dots = \sum_{k=1}^{\infty} k$$

A **partial sum** (of a series) is the sum of part of a series

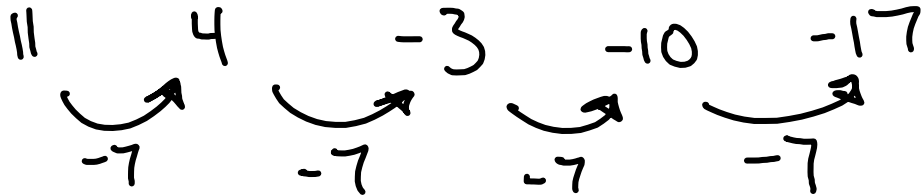
$$1 + 2 + 3 + 4 = \sum_{k=1}^4 k = 10$$

Arithmetic Sequence / Series: What it is (and how to identify it)

An arithmetic sequence's first difference (next term - current term) is constant:



To test if a sequence is arithmetic, compute first difference. If its constant then sequence is arithmetic.



Arithmetic Series / Partial Sum: What do they look like in summation notation?

Example:

$$10 + 12 + 14 + 16 + \dots = \sum_{k=0}^{\infty} 10 + 2k$$

Diagram illustrating the arithmetic series $10 + 12 + 14 + 16 + \dots$ and its summation notation $\sum_{k=0}^{\infty} 10 + 2k$. The first four terms are shown with arrows indicating the common difference of 2. The terms are labeled as $10+2\cdot 0$, $10+2\cdot 1$, $10+2\cdot 2$, and $10+2\cdot 3$. The summation notation shows the starting value 10 and the index k .

$$a_k = 10 + 2k$$

$$a_0 \quad a_1 \quad a_2 \quad a_3$$

Every arithmetic series can be expressed via the following form:

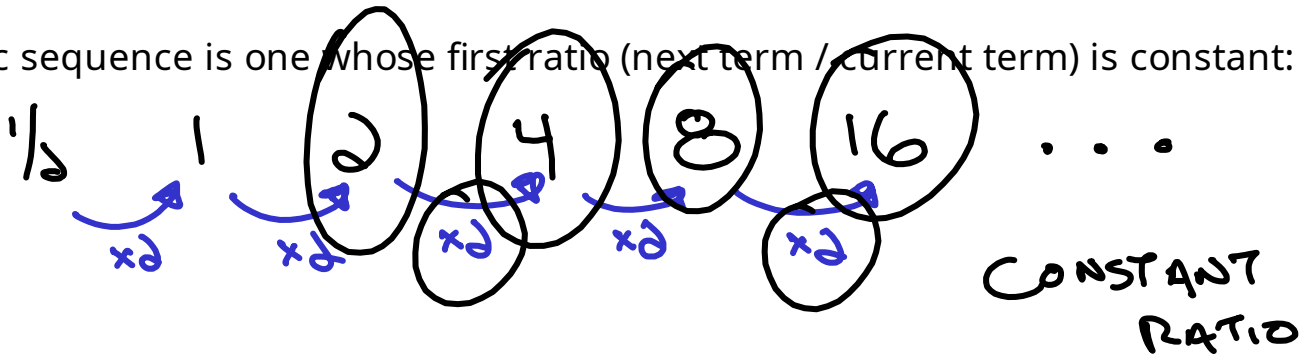
$$\sum_{k=0}^{\infty} a_0 + dk$$

Diagram illustrating the general form of an arithmetic series summation: $\sum_{k=0}^{\infty} a_0 + dk$. The components are labeled:

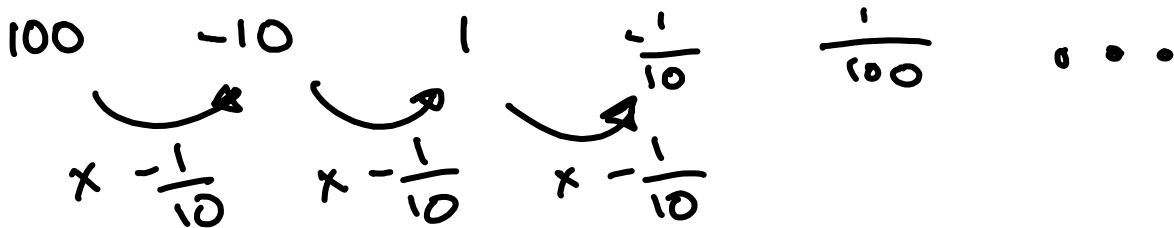
- STARTING VALUE**: a_0
- INDEX**: k
- DIFFERENCE BETWEEN ADJACENT VALUES**: d

Geometric Sequences / Series: What it is (and how to identify it)

An Geometric sequence is one whose first ratio (next term / current term) is constant:



To test if a sequence is geometric, compute first ratio. If its constant then sequence is geometric.



Geometric Series / Partial Sum: What do they look like in summation notation?

Example:

$\frac{1}{2} + 1 + 2 + 4 + 8 + \dots = \sum_{k=0}^{\infty} \frac{1}{2} \cdot 2^k$

$a_k = a_0 \cdot r^k$

Every geometric series can be expressed via the following form:

$\sum_{k=0}^{\infty} a_0 \cdot r^k$

- STARTING TERM
- RATIO OF NEXT TERM / CURRENT TERM
- INDEX

Quadratic Series / Partial Sum: What is it? (i.e. what does it look like in sum notation?)

Every quadratic series can be expressed as:

$$\sum_{k=0}^{\infty} a \underline{k^2} + b \underline{k} + c$$

FIRST TERM

a, b, c ARE CONSTANT
(NOT A EASILY SEEN AS
ARITHMETIC / GEOMETRIC)

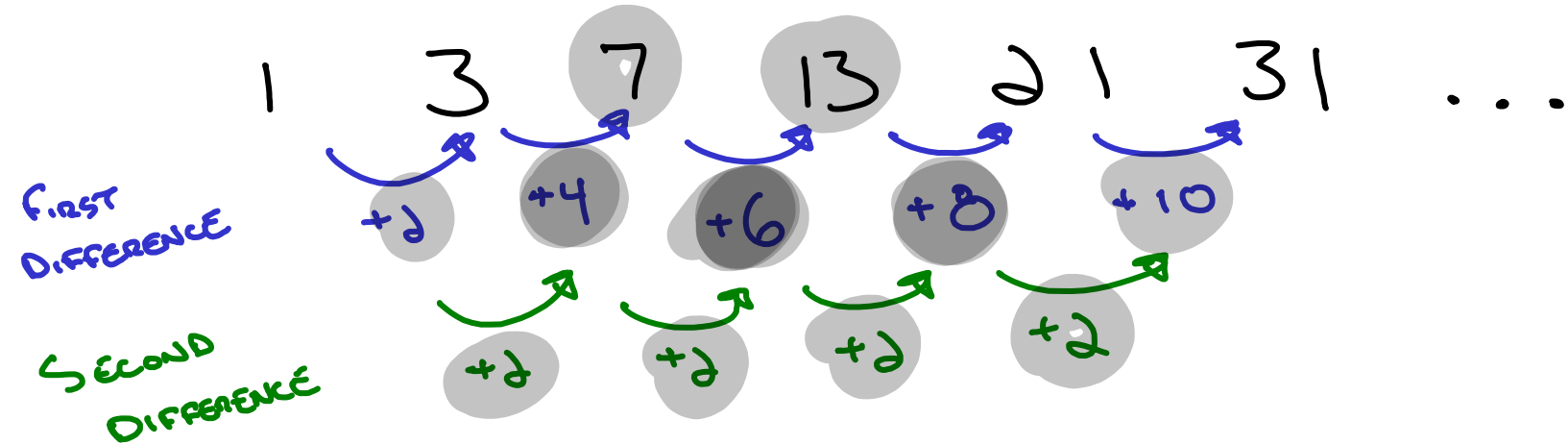
Example (a=1, b=0, c=0):

$$\begin{array}{ccccccc} & & & & 1 \cdot 4^2 + 0 \cdot 4 + 0 & & \\ & & & & \downarrow & & \\ & & & & 16 & + & 25 + \dots \\ & & & & \uparrow & & \\ & & & & 1 \cdot 5^2 + 0 \cdot 5 + 0 & & \\ & & & & \uparrow & & \\ & & & & 25 & + & \dots \\ & & & & \uparrow & & \\ & & & & 1 \cdot 3^2 + 0 \cdot 3 + 0 & & \\ & & & & \uparrow & & \\ & & & & 9 & + & 16 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 2^2 + 0 \cdot 2 + 0 & & \\ & & & & \uparrow & & \\ & & & & 4 & + & 9 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 1^2 + 0 \cdot 1 + 0 & & \\ & & & & \uparrow & & \\ & & & & 1 & + & 4 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 1 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 0 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 0 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 0 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 0 \\ & & & & \uparrow & & \\ & & & & 1 \cdot 0^2 + 0 \cdot 0 + 0 & & \\ & & & & \uparrow & & \\ & & & & 0 & + & 0 \end{array}$$

Question (for later): given the first few values in sequence, how can we get a, b, c?

Quadratic Sequences / Series: How to identify it

The second difference of a quadratic sequence is constant



In Class Activity:

Identify the type (arithmetic, geometric, quadratic, or none) of each of the following sequences. If sequence is arithmetic or geometric, express its corresponding series in sum notation.

i. **QUADRATIC**
6, 15, 28, 45, 66, 91, ...
+9 +13 +17 +21 +25
+4 +4 +4 +4

iii. 4, 7, 10, 13, 16, 19, ...
+3 +3 +3 +3

CONSTANT 1st DIFF \rightarrow ARITHMETIC
$$\sum_{k=0}^{\infty} 4 + 3k$$

ii. **GEOMETRIC**
1, -4, 16, -64, 256, ...
 $x-4$ $x-4$ $x-4$
$$\sum_{k=0}^{\infty} 1 \cdot (-4)^k$$

iv. 2, 7, 11, 42, -4, ...

Not ARITHMETIC / GEOMETRIC
OR QUADRATIC

In Class Activity:

Identify the type (arithmetic, geometric, quadratic, or none) of each of the following sequences. If sequence is arithmetic or geometric, express its corresponding series in sum notation.

i. 6, 15, 28, 45, 66, 91, ... **QUADRATIC**

$$\begin{array}{ccccccc} & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ & 9 & 13 & 17 & 21 & & \\ + & & & & & & \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \\ & 4 & 4 & 4 & 4 & & \end{array}$$

ii. 1, -4, 16, -64, 256, ... **GEOMETRIC**

$$\begin{array}{ccccccc} & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ & -4 & \times -4 & \times -4 & \times -4 & & \\ + & & & & & & \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \\ & 1 & -4 & 16 & -64 & 256 & \end{array}$$

$$\sum_{k=0}^{\infty} (-4)^k$$

iii. 4, 7, 10, 13, 16, 19, ... **ARITHMETIC**

$$\begin{array}{ccccccc} & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ & 3 & 3 & 3 & 3 & & \\ + & & & & & & \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \\ & 4 & 7 & 10 & 13 & 16 & 19 \end{array}$$

$$\sum_{k=0}^{\infty} 4 + 3k$$

iv. 2, 7, 11, 42, -4, ... **NONE**

$$\begin{array}{ccccccc} & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \\ & 5 & 4 & 31 & & & \\ + & & & & & & \\ & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \nwarrow & \\ & 2 & 7 & 11 & 42 & -4 & \end{array}$$

Quadratic Series: Given sequence, how to compute a, b, c in summation notation

$$6 + 15 + 28 + 45 + 66 + 91 + \dots$$

$k=0$ points to 6, $k=1$ points to 15, $k=2$ points to 28, $k=3$ points to 45, $k=4$ points to 66, $k=5$ points to 91.

$$= \sum_{k=0}^{\infty} ak^2 + bk + c$$

$$6 = a \cdot 0^2 + b \cdot 0 + c \rightarrow c = 6$$

$$9 = 2 + b \quad b = 7$$

$$15 = a \cdot 1^2 + b \cdot 1 + c \rightarrow 15 = a + b + 6 \rightarrow 9 = a + b \rightarrow b = 9 - a$$

$$28 = a \cdot 2^2 + b \cdot 2 + c \rightarrow 28 = 4a + 2b + 6 \rightarrow 22 = 4a + 2b$$

$$\rightarrow 11 = 2a + b$$

$$11 = 2a + 9 - a$$

$$2 = a$$

Quadratic Series: Given sequence, how to compute a, b, c in summation notation

$$\begin{array}{ccccccc} k=0 & & k=2 & & k=4 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 6 & + & 15 & + & 28 & + & 45 & + & 66 & + & 91 & + & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & & & & & \\ & & k=1 & & k=3 & & k=5 & & & & & & \end{array}$$

$$= \sum_{k=0}^{\infty} ak^2 + bk + c$$

$$6 = a \cdot 0^2 + b \cdot 0 + c$$

$$c = 6$$

$$15 = a \cdot 1^2 + b \cdot 1 + c$$

$$\rightarrow 15 = a + b + 6$$

$$\begin{aligned} 9 &= a + b \\ b &= 9 - a \end{aligned}$$

$$28 = a \cdot 2^2 + b \cdot 2 + c$$

$$\rightarrow 28 = 4a + 2b + 6$$

$$22 = 4a + 2b$$

$$11 = 2a + b$$

$$\begin{aligned} 9 &= a + b \\ 7 &= 2a + b \end{aligned}$$

$$11 = 2a + 9 - a$$
$$2 = a$$

Checking our work with python

(you needn't ever do the same for CS1800 ... but cute to see that you can using python)

```
matt@matt-yoga-nu:~$ python3
Python 3.10.12 (main, Jun 11 2023, 05:26:28) [GCC 11.4.0] on linux
Type "help", "copyright", "credits" or "license" for more information.
>>> a, b, c = 2, 7, 6
>>> [a * k ** 2 + b * k + c for k in range(10)]
[6, 15, 28, 45, 66, 91, 120, 153, 190, 231] → SAME AS
>>> █                               GIVEN 😊
```

If you're interested in doing the same and don't have python on your computer, check out "google colab" which allows you to run python code in the cloud.

Quadratic Series Convention: start counting at k=0 or k=1?

$$\begin{array}{ccccccc} \text{K=0} & & \text{K=2} & & \text{K=4} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 6 & + & 15 & + & 28 & + & 45 & + & 66 & + & 91 & + & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & & & & & \\ & & \text{K=1} & & \text{K=3} & & \text{K=5} & & & & & & \end{array}$$

$$6 = a \cdot 0^2 + b \cdot 0 + c$$

$$15 = a \cdot 1^2 + b \cdot 1 + c$$

$$28 = a \cdot 2^2 + b \cdot 2 + c$$

$$\begin{array}{ccccccc} \text{K=1} & & \text{K=3} & & \text{K=5} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 6 & + & 15 & + & 28 & + & 45 & + & 66 & + & 91 & + & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & & & & & \\ & & \text{K=2} & & \text{K=4} & & \text{K=6} & & & & & & \end{array}$$

$$6 = a \cdot 1^2 + b \cdot 1 + c$$

$$15 = a \cdot 2^2 + b \cdot 2 + c$$

$$28 = a \cdot 3^2 + b \cdot 3 + c$$

Starting to count at k=0 (left) or k=1 (right) yields different a,b,c. Both are correct in their own contexts. Prefer starting to count at k=0 (left), its easier: that first equation immediately gives c.

In Class Activity

Find the coefficients (a, b, c) which allow us to express the following series in summation notation (convention: first term has $k=0$)

$$1 + 3 + 7 + 13 + 21 + 31 + 43 + 57 + 73 + 91 + \dots$$

$$\sum_{k=0}^{\infty} ak^2 + bk + c$$

$$1 = a \cdot 0^2 + b \cdot 0 + c \rightarrow c = 1$$

$$3 = a \cdot 1^2 + b \cdot 1 + c \rightarrow 3 = a + b + 1 \rightarrow b = 2 - a$$

$$7 = a \cdot 2^2 + b \cdot 2 + c \rightarrow 7 = 4a + 2b + 1 \rightarrow 6 = 4a + 2b$$

$$3 = 2a + b$$

$$3 = 2a + 2 - a$$

$$1 = a$$

$$b = 2 - a \\ = 2 - 1 = 1$$

In Class Activity

Find the coefficients (a, b, c) which allow us to express the following series in summation notation (convention: first term has $k=0$)

$$1 + 3 + 7 + 13 + 21 + 31 + 43 + 57 + 73 + 91 + \dots = \sum_{k=0}^{\infty} ak^2 + bk + c$$

(Note: Handwritten arrows under the first four terms of the series point to the differences 2, 4, 6, 8.)

$$1 = 0^2 \cdot a + 0 \cdot b + c \rightarrow c = 1$$

$$3 = 1^2 \cdot a + 1 \cdot b + c \rightarrow 3 = a + b + 1 \rightarrow b = 2 - a$$

$$7 = 2^2 \cdot a + 2 \cdot b + c \rightarrow 7 = 4a + 2b + 1 \rightarrow 6 = 4a + 2(2 - a) \\ = 4a + 4 - 2a$$

$$b = 2 - a \\ = 2 - 1 = 1$$

$$2 = 2a \\ 1 = a$$

Up next: computing partial sums (arithmetic & geometric ... not quadratic)

ARITHMETIC

$$0 + 1 + 2 + 3 + 4 = \sum_{k=0}^4 k = ?$$

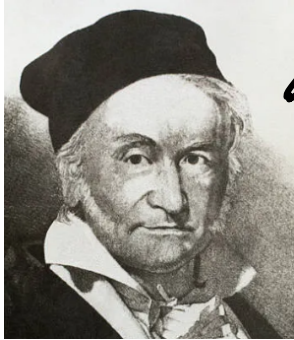
GEOMETRIC

$$1 + 2 + 4 + 8 + 16 = \sum_{k=0}^4 2^k = ?$$

↓
NO SIMPLE
FORMULA
EXISTS ☹️

Computing Arithmetic Partial Series: motivation via tall tale

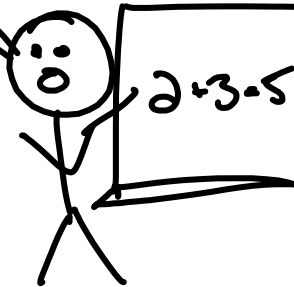
PRIMARY
SCHOOL
GAUSS



Gauss, you're not paying attention. As punishment go in the hall and add all the integers from 1 to 100

Its 5050

TEACHER



$$0 + 1 + 2 + \dots + 98 + 99 + 100$$

$$2 + 98 = 100$$

$$1 + 99 = 100$$

$$0 + 100 = 100$$

50 sums of 100
+ LEFTOVER 50 = 5050

Computing Arithmetic Sums: A more generalizable expression

SMALL TEST EXAMPLE

$$1 + 2 + 3 + 4 + 5 = 15$$

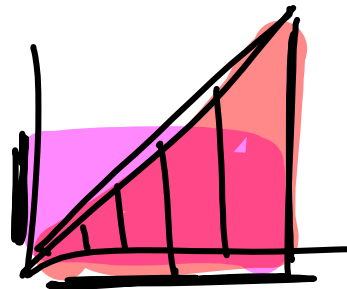
Average Term

x Number of Terms

$$\frac{1+5}{2} = 3$$

x

5



Notice:

Summing from $k=0$ up to N has $N + 1$ total terms.

$$\sum_{k=0}^N a_0 + dk = \left(\frac{a_0 + a_N}{2} \right) \times (N+1)$$

(should we choose convention that our first term is $k=1$, then this formula changes a bit to have N total terms)

Computing Arithmetic Sums: A more generalizable expression

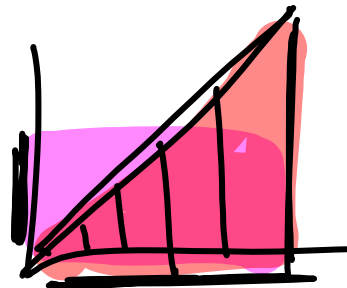
SMALL TEST EXAMPLE

$$0 + 1 + 2 + 3 + 4 + 5 = 15$$

Average Term \times Number of Terms

$$\frac{0+5}{2} = 2.5 \quad 6$$

$$\sum_{k=0}^N a_0 + dk = \left(\frac{a_0 + a_N}{2} \right) \times (N+1)$$



Notice:

Summing from $k=0$ up to N has $N+1$ total terms.

(should we choose convention that our first term is $k=1$, then this formula changes a bit to have N total terms)

Computing Geometric Series Partial Sums

S is the
PARTIAL SUM
WE'D LIKE TO
COMPUTE

$$S = \sum_{k=0}^N ar^k = a + ar + ar^2 + \dots + ar^N$$

$$S \cdot r = ar + ar^2 + \dots + ar^N + ar^{N+1}$$

$$S - S \cdot r = a - ar^{N+1}$$

$$\text{so } S(1-r) = a(1-r^{N+1}) \Rightarrow$$

$$S = \frac{a(1-r^{N+1})}{1-r}$$

Computing Geometric Series: Lets work a little example to check if that formula works

$$1 + 2 + 4 + 8 + 16 = \sum_{k=0}^4 1 \cdot 2^k = 31$$

Diagram illustrating the sum of a geometric series. The terms 1, 2, 4, 8, and 16 are shown, with arrows indicating the common ratio of 2 between consecutive terms. The terms are labeled with $k=0$ through $k=4$. The sum is equated to the summation formula $\sum_{k=0}^4 1 \cdot 2^k = 31$.



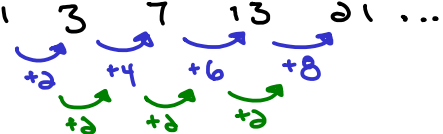
$$S = \frac{a_0 (1 - r^{N+1})}{1 - r} = \frac{1 \cdot (1 - 2^5)}{1 - 2} = \frac{-31}{-1} = 31$$

Diagram illustrating the formula for the sum of a geometric series. The first term a_0 is highlighted in red. The common ratio r is highlighted in orange. The number of terms $N+1$ is highlighted in yellow. The formula is shown as $S = \frac{a_0 (1 - r^{N+1})}{1 - r}$. The values are substituted as $S = \frac{1 \cdot (1 - 2^5)}{1 - 2} = \frac{-31}{-1} = 31$.


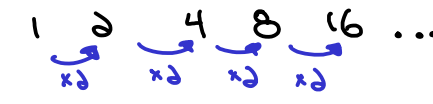
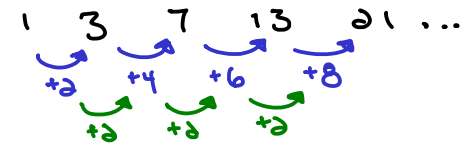
Labels:

- First TERM
- $r = \text{RATIO}$
- $N = \text{LARGEST VALUE OF } K \text{ IN SUM}$

In summary (Arithmetic, Geometric & Quadratic Sequences / Series / Partial Sums)
 (assumes first term is $k=0$)

	Arithmetic	Geometric	Quadratic
How to identify?	 <p>CONSTANT FIRST DIFFERENCE</p>	 <p>CONSTANT RATIO</p>	 <p>CONSTANT SECOND DIFFERENCE</p>
Expression of a single term	$a_0 + dK$	$a_0 r^K$	$aK^2 + bK + c$
Computing partial sum	$\sum_{k=0}^N a_0 + dK = \left(\frac{a_0 + a_N}{2} \right) \cdot (N+1)$ <p style="text-align: center;"> ↑ AVERAGE TERM ↑ NUMBER OF TERMS </p>	$\sum_{k=0}^N a_0 r^K = \frac{a_0 (1 - r^{N+1})}{1 - r}$	KIND OF A CALCULUS THING (NOT NEEDED FOR CS1800)

In summary (Arithmetic, Geometric & Quadratic Sequences / Series / Partial Sums)
 (assumes first term is k=1)

	Arithmetic	Geometric	Quadratic
How to identify?	<p>2 4 6 8 ...</p>  <p>CONSTANT FIRST DIFFERENCE</p>	<p>1 2 4 8 16 ...</p>  <p>CONSTANT RATIO</p>	<p>1 3 7 13 21 ...</p>  <p>CONSTANT SECOND DIFFERENCE</p>
Expression of a single term	$a_1 + d(k-1)$	$a_1 r^{k-1}$	$ak^2 + bk + c$
Computing partial sum	$\sum_{k=1}^N a_1 + d(k-1) = \left(\frac{a_1 + a_N}{2} \right) \cdot N$ <p style="text-align: center;"> ↑ AVERAGE TERM ↑ NUMBER OF TERMS </p>	$\sum_{k=1}^N a_1 r^{k-1} = \frac{a_1 (1 - r^N)}{1 - r}$	<p>KIND OF A CALCULUS THING (NOT NEEDED FOR CS1800)</p>

In Class Activity:

Compute each of the following sums (using the partial sums formula)

i.
$$\sum_{k=0}^{100} 4 - 1k = \left(\frac{a_0 + a_N}{d} \right) (N+1) = \left(\frac{4 + 4 - 100}{d} \right) (101) = -46 \cdot 101 = -4646$$

ii.
$$\sum_{k=0}^{10} 10 \cdot 3^k = \frac{a_0(1-r^{N+1})}{1-r} = \frac{10(1-3^{11})}{1-3} = 885730$$

iii.
$$10 + 7 + 4 + 1 + (-2) + (-5) + (-8)$$

$$\sum_{k=0}^6 10 - 3k = \left(\frac{a_0 + a_N}{d} \right) (N+1) = \left(\frac{10 - 8}{d} \right) (7) = 7$$

In Class Activity:

Compute each of the following sums (using the partial sums formula)

i.
$$\sum_{k=0}^{100} 4 - 1k = \left(\frac{a_0 + a_N}{2} \right) \cdot (N+1) = \frac{4 + (4 - 100)}{2} \cdot 101$$

ii.
$$\sum_{k=0}^{10} 10 \cdot 3^k = \frac{a_0(1 - r^{N+1})}{1 - r} = \frac{10(1 - 3^{11})}{1 - 3}$$

iii. $10 + 7 + 4 + 1 + (-2) + (-5) + (-8)$

$$\sum_{k=0}^6 10 - 3k = \left(\frac{a_0 + a_N}{2} \right) \cdot (N+1) = \frac{10 - 8}{2} \cdot 7$$