Professor Hamlin
Day 8

Agenda:
→ Regrades → Gradescope
→ HW1 formatting grade reduced by portortional HW3
30% on HW1
15% on HW3
formatting deduction reduced by 1/2

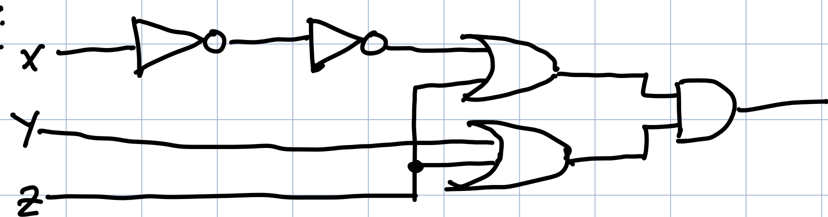
1) Admin
2) Review
3) Pigeonhole Principle
4) Counting
→ Product rule → cartesian product
→ sum rule → inclusion & exclusion

Review:

sets: represent on computer as bitstring
logic = set op   AND / Intersection

algebra: simplifying expression
circuits: wires, gates: AND, OR, NOT, XOR

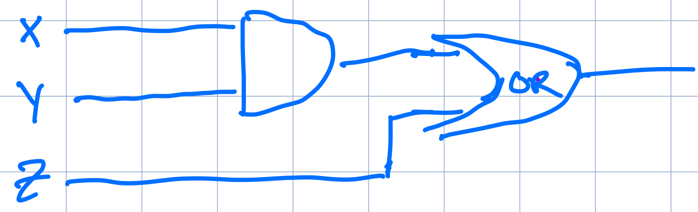
Exercise:



1) write boolean expression
$(\neg\neg x \vee z) \wedge (y \vee z)$

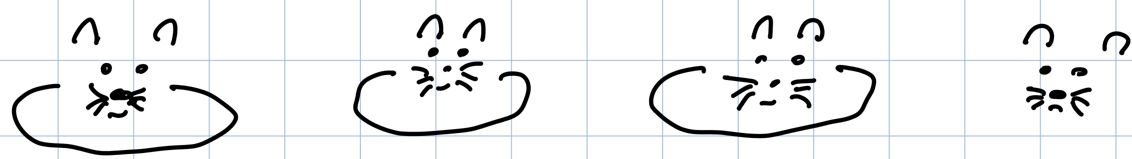
2) simplify expression
$(x \vee z) \wedge (y \vee z)$ double negative
$z \vee (x \wedge y)$ distributive

3) redraw circuit of simplified expression



Pigeonhole principle

(imagine) I have 3 cat beds and 4 cats



How can cats be distributed on the beds?

B1 C1
B2 C2 C4
B3 C3
~ or ~
B1 C1 C2 C3 C4
B2
B3

$\exists$ a cat bed with at least 2 cats

But w/ only # of beds & cats don't know
1) that all beds will have cats on them
2) or the exact # of cats on each bed

We can say if we have 4 (N) cats and 3 (K) beds then at least one bed will have at minimum $\lceil 4/3 \rceil$ cats

$\lceil N/K \rceil$ ← ceiling, when dividing always round up

$\lceil 6/4 \rceil = \lceil 1.5 \rceil = 2$ $\lfloor 6/4 \rfloor = \lfloor 1.5 \rfloor = 1$ ← floor

$\lceil 5/4 \rceil = \lceil 1.25 \rceil = 2$

Let's try this out ~ 3 beds $(N=3)$

$(K=)$ Number of cats	0	1	2	3	4	5	6	7	8	9	10	11
$(\lceil N/K \rceil)$ guarenteed minimum of cats on one bed	–	1 $\lceil 1/3 \rceil$	1 $\lceil 2/3 \rceil$	1	2 $\lceil 4/3 \rceil$	2 $\lceil 5/3 \rceil$	2	3 $\lceil 7/3 \rceil$	3	3	4 $\lceil 10/3 \rceil$	4

Pigeonhole Principle if we divide N items into K piles there exists a pile with a minimum of $\lceil N/K \rceil$ items

$$x = \lceil N/K \rceil$$

N → # of cats

K → # of beds

x → ∃ bed with at least x cats in it

<u>Exercise</u>

1) We have 3 pigeons and 2 nests. How many pigeons, at minimum, will be on at least one nest?

$N=3$ $\quad k=2$ $\quad \lceil 3/2 \rceil = \boxed{2}$

2) If we group people in class by birth month how many people will be in the largest group at minimum? # of people = 149

$N=149$ $\quad K=12$

$$\left\lceil \frac{149}{12} \right\rceil = 13$$

Say we want to publish exam grades online.

- Each student has a 2-digit secret hex code
- If you know it, can look up grade
- otherwise anonymous

Code	C1	DF	19	9F	B2	24	
Grade	A-	A	B+	B	A	A-	....

How many students can we support w/ 2-digits of hex? (We don't want collisions!)

N = # of students

k = # of hex codes

i.e. $1 = \lceil N/k \rceil$

What is k? $\underline{0\text{-}F}$ $\underline{0\text{-}F}$

256

smallest 00 0

largest FF → 255

$100_{16} - 1$

$16^2 - 1$

So N can't be any bigger than that!

N = 258 $\left\lceil \frac{258}{256} \right\rceil = 2$

N = 256 $\left\lceil \frac{256}{256} \right\rceil = 1$

$\boxed{N \leq 256}$

For a class of 800 we will need more Hex digits!

How many? – Counting

16^2 16^3

Counting

A computer can try and guess a password 1000 times a second* how will a 4 digit passcode hold up

- 4 numbers?   1532, 9875 .... $\underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 10^4$ $\boxed{10s}$
- 4 letters? $\underline{26}\cdot\underline{26}\cdot\underline{26}\cdot\underline{26} = 26^4$ $\boxed{457s}$

What about longer
- 8 numbers? $\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10}\cdot\underline{10} = 10^8$ $\boxed{10^5 s}$
- 8 letters? $26\cdot 26 \ldots 26 = 26^8$ $\boxed{2.1\times 10^9 s}$

*closer to 14 billion per second

Longer passwords are better than ones w/ more characters!

Product rule

Getting dressed in the morning

shirt  3 shirts

~and~

pants  2 pants

~and~

socks  2 pairs of socks

How many different outfits can I wear?

Shirts = {A, B, C}
pants = {1, 2}
Socks = {a, b}

(A, 1, a)
(A, 1, b)
⋮
(C, 2, b)

We can capture this formally - cartesian product
set of elements in A paired w/ all of elements in B

$A = \{1, 2, 3\}$ $B = \{1, 2\}$

$A \times B = \{ (1,1), (1,2), (2,1), (2,2), (3,1), 3,2) $

Tuple: may repeat, order matters
$(1,2) \neq (2,1)$

crossproduct = cartesian product

$$\boxed{\text{Note} \quad A \times B \neq B \times A \rightarrow \{(1,1)\ (1,2), (1,3) \ldots}$$

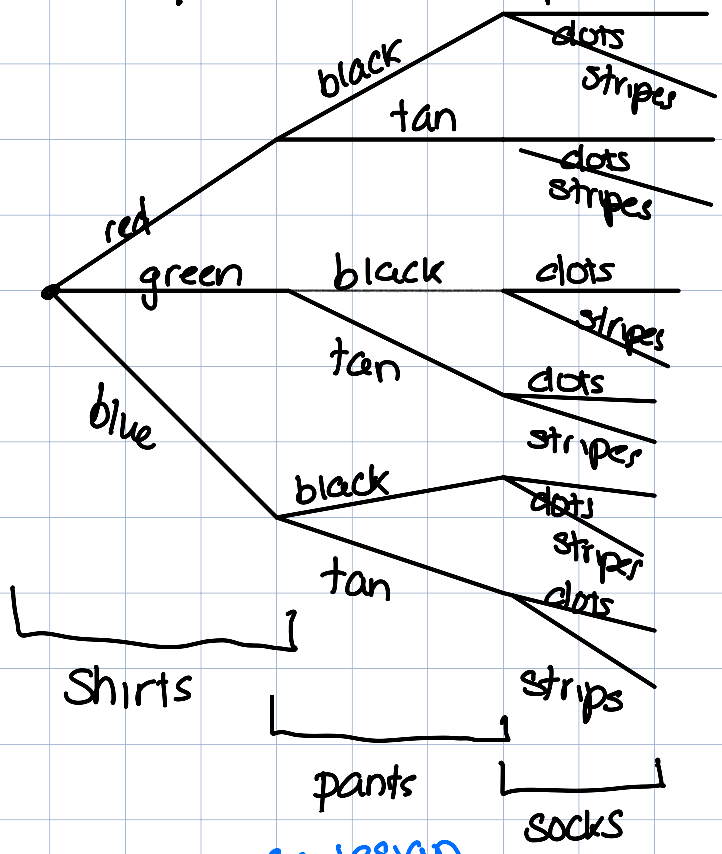
Exercise: What is cartesian product of
shirts = {red, blue}   pants = {black, tan}
socks = {dots, stripes}

R B B T D S

shirts x pants x socks = { (R, B, D), (R, B, S), (R, T, D), (R, T, S), (B, B, D), (B, B, S), (B, T, D), (B, T, S)}

$\boxed{\text{8 outfits}}$

How many w/ 3 shirt options? {red, green, blue}



shirts x pants x socks has...

$3 \times 2 \times 2$
12 outfits

4 different pairs of socks...

$3 \times 2 \times 4 = 24$ outfits

Formally: cartesian, multi, cardinality

$|A \times B| = |A| * |B|$

$|A \times B \times C| = |A| * |B| * |C|$

$\vdots$

$\boxed{\text{Product rule is the cardinality of cartesian product of the set options}}$

We use product rule when we have an 'and' between choices.

shirt AND pants AND socks

not

shirt OR dress (cover shortly)

Exercise 1) How many passwords of length 4 can be made w/ 4 lower case letters? L = set of lower case letter

$$|L \times L \times L \times L| = |L| * |L| * |L| * |L|$$
$$= 26 * 26 * 26 * 26 = \boxed{26^4}$$

2) ... upper & lowercase letters A = set of upper & lower

$$|A \times A \times A \times A| = |A| * |A| * |A| * |A|$$
$$52 \cdot 52 \cdot 52 \cdot 52 = \boxed{52^4}$$

3) ... if 1st letter is 'a' and the rest are lowercase?

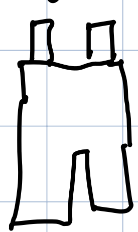
$$|\{a\} \times L \times L \times L| = |\{a\}| * 26 * 26 * 26$$
$$1 \cdot \boxed{26^3}$$

4) if 1st letter is 'a', 'b', or 'c', and rest are lower case?

$$|\{a, b, c\} \times L \times L \times L| = |\{a, b, c\}| * 26 * 26 * 26$$
$$\boxed{3 \cdot 26^3}$$

Sum Rule

Getting dressed - overalls or dress



overalls = {denim, corderoy}

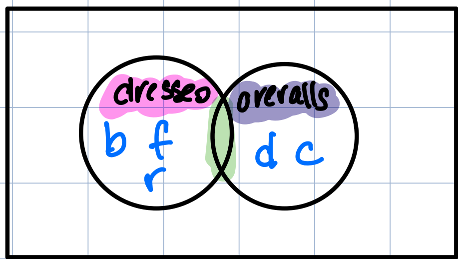
~or~

dress → {blue, flowers, red}

How many options? $\boxed{5}$

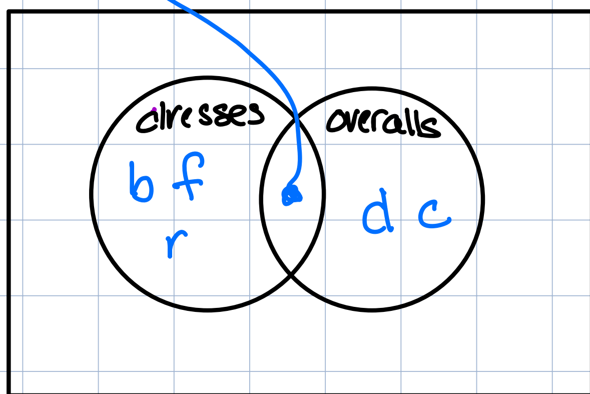
no overlap

Formally if the sets A & B are disjoint the items in $|A \cup B| = |A + |B|$



$$|A \cup B| = |A| + |B|$$

But what about .... the overall dress??

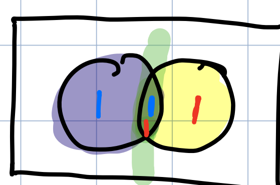


$$|A \cup B| \neq |A| + |B|$$
$$\neq \quad 4 + 3$$
$$6 \neq 7$$

$A \cap B$ is counted twice!

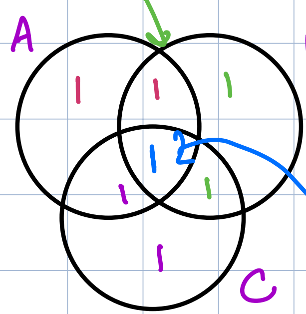
Principle of Inclusion-exclusion (PIE): when counting union # items in A + items in B minus any in the intersection double counted

$$|A \cup B| = |A| + |B| - |A \cap B|$$



PIE for 3 sets:

double counted



removed three times

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$|M \cup S \cup C| = 17$$

Example: 17 employees w/ 3 different roles (manage, stock, checkout) — M, S, C

- 3 trained as managers $|M| = 3$
- 10 trained to stock shoes $|S| = 10$
- 7 trained at checkout $|C| = 7$
- 1 employee has double training in each pair of jobs $|M \cap S| = |M \cap C| = |S \cap C| = 1$

How many employees are trained to do all three jobs? $|M \cap S \cap C| = ?$

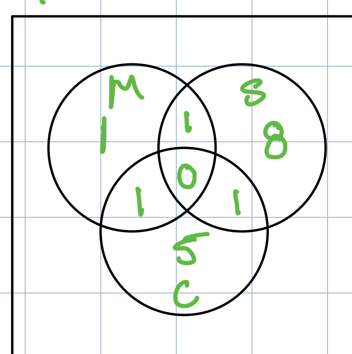
$$|M \cup S \cup C| = |M| + |C| + |S| - |M \cap S| - |M \cap C| - |S \cap C| + |M \cap S \cap C|$$

?

$$17 = 3 + 10 + 7 - 1 - 1 - 1 + x$$

$$\boxed{x = 0}$$

$$|M \cap S \cap C| = 0$$



Exercise:

Of the 196 kindergarden students which like gym or music or art:

G M A

45 like gym class
90 like music class
100 like art class
20 like both gym and music
13 like both gym and art
7 like both art and music

$|G \cup M|$

1) how many students like gym or music?
2) how many students like all 3 subjects? $|G \cap M \cap A|$
~~3) how many students like gym but nothing else~~

$|G \cup M \cup A| = 196$

$|G| = 45$

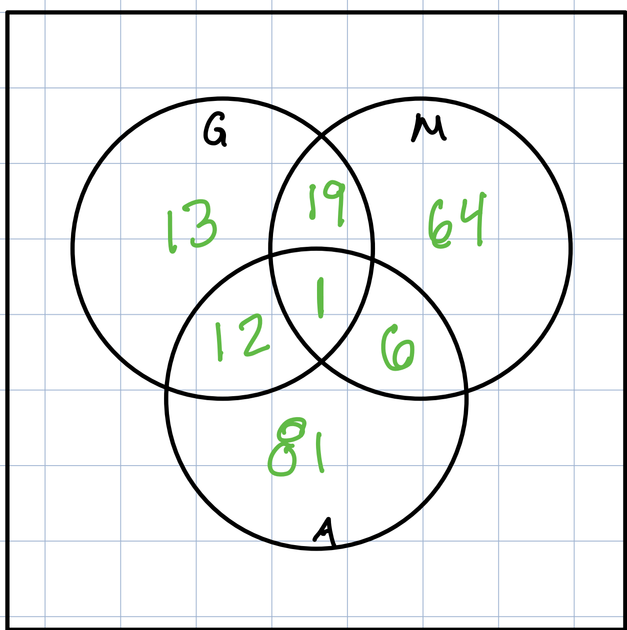
$|M| = 90$

$|A| = 100$

$|G \cap M| = 20$

$|G \cap A| = 13$

$|A \cap M| = 7$



1) $|G \cup M| = |G| + |M| - |G \cap M|$

$45 + 90 - 20$

$= \boxed{115}$

2) $|G \cap M \cap A| = ?$

$|G| + |A| + |M| - |G \cap M| - |G \cap A| - |A \cap M| + \underline{|G \cap M \cap A|} = |G \cup M \cup A|$

$196 = 45 + 90 + 100 - 20 - 13 - 7 + x$

$x = 1$