CS1800 Day 12

Admin:

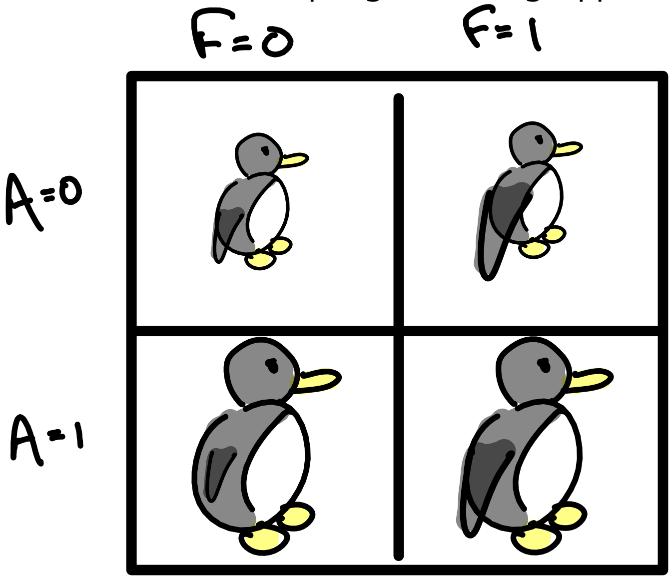
- exam / HW4 results & grade estimates (available by Friday afternoon, likely soon)
- tuning up your study process in CS1800

Content:

Joint Probability Distribution
Marginalization
Conditional Probability
Bayes Rule
Independence

Joint Probability Distribution: A distribution over more than 1 variable at a time

Let A=1 indicate if a penguin is an adult (0 otherwise)
Let F=1 indicate if a penguin has big flippers (0 otherwise)



	F=0	F=1
A=0	3/12	2/12
A=1	1/12	6/12

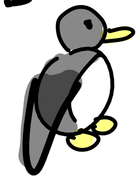
HALF OF PENGUINS ARE ADULTS w/ BIG FLIPPERS

NOTATION

$P(A=0, F=1)$ IS PROB OF BOTH OUTCOMES OCCURING AT SAME TIME

$A=0$ (NOT ADULT)

$F=1$ (BIG FLIPPER)

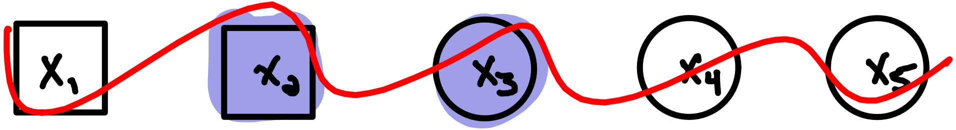


Marginalizing (removing a random variable from a joint distribution)

Consider selecting one of the following 5 shapes, with equal probability:

Let B=1 be the event that the shape is blue

Let C=1 be the event that the shape is a circle



	B=0	B=1
C=0	1/5	1/5
C=1	2/5	1/5

WHAT IS $P(B=1) = P(B=1\ C=0) + P(B=1\ C=1)$

$= 1/5 + 1/5$

$= 2/5$

Remember: To compute P(B) we can sum P(B, A) for all outcomes in sample space of A

$$P(B=b) = \sum_a P(B=b, A=a)$$

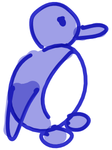
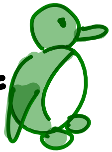
In Class Activity

Let C be a random variable representing penguin color (sample space: blue, red or green)
Let A=1 indicate if a penguin is an adult (0 otherwise)

Given the following distribution of A, C

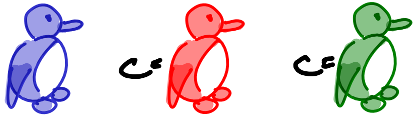
Compute each of the follow probabilities:

- P(C=blue)
- P(C=red) + P(C=green)
  (how is this related to prob above?)
- P(A=1)

	C= 	C= 	C= 
A=0	1/12	3/12	0/12
A=1	2/12	1/12	5/12

In Class Activity

C= C= C=



A=0	1/12	3/12	0/12
A=1	2/12	1/12	5/12

- P(C=blue) $= \sum_a P(C=\text{BLUE}\ A=a)$

$$= P(C=\text{BLUE}\ A=0) + P(C=\text{BLUE}\ A=1)$$

$$1/12 + 2/12 = 3/12$$

- P(C=red) + P(C=green)

  (how is this related to prob above?)

$$P(C=\text{RED}) = 3/12 + 1/12 = 4/12$$

$$P(C=\text{GREEN}) = \frac{0}{12} + \frac{5}{12} = 5/12$$

$$P(C=\text{RED OR GREEN}) = 9/12$$

- P(A=1)

$$P(A=1) = P(A=1\ C=\text{BLUE}) + P(A=1\ C=\text{RED}) + P(A=1\ C=\text{GREEN})$$

$$= 2/12 + 1/12 + 5/12 = \frac{8}{12}$$

Conditional Probability (intuition & motivation)

$C=1$ indicates a person has covid ($C=0$ otherwise)
$T=1$ indicates a person has positive test ($T=0$ otherwise)
Let us discuss (and express) the following probabilities:

- probability person has a positive test

$P(T=1)$ is the fraction of EVERYONE who is testing positive

- probability person has positive test given they have covid

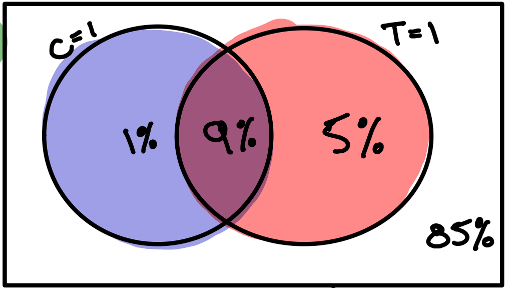
$P(T=1|C=1)$ is the fraction of THOSE WHO HAVE COVID who are testing positive

- probability person has covid given a positive test

$P(C=1|T=1)$ is the fraction of THOSE WHO ARE TESTING POSITIVE who have covid

Intuition:
Conditional probability $P(X=x|Y=y)$ is the probability of event $X=x$ if we constrain ourselves to a world where $Y=y$.

Conditional Probability (motivating our formula from intuition)



C=1 indicates a person has covid (C=0 otherwise)
T=1 indicates a person has positive test (T=0 otherwise)

Let us discuss (and express) the following probabilities:

- probability person has a positive test $P(T=1) = \underset{=}{P}(T=1\ \ C=0) + P(T=1\ \ C=1) = 5 + 9 = 14\%$

- probability person has positive test given they have covid $P(T=1 \mid C=1) = \frac{9}{1+9} = 90\%$

- probability person has covid given a positive test

$$P(C=1 \mid T=1) = \frac{9}{9+5} = \frac{P(C=1\ \ T=1)}{P(C=1\ \ T=1) + P(C=0\ \ T=1)} = \frac{P(C=1\ \ T=1)}{P(T=1)}$$

Intuition:
Conditional probability $P(X=x|Y=y)$ is the probability of event $X=x$ if we constrain ourselves to a world where $Y=y$.

Conditional Probability (Formula version1: from our intuition)

$$P(a|b) = \frac{P(a\,b)}{P(b)}$$

PROB a HAPPENS GIVEN CONDITION b

PROB a b HAPPEN TOGETHER

PROB b HAPPENS

In Class Activity

Compute each of the probabilities from the distribution below. For each, write a sentence explaining its meaning which is easily understood by a non-technical reader

Let S be a twitter sentiment score about bitcoin (1=good, 0=neutral, -1=bad)
Let B be the movement of bitcoin price (1=up, -1=down)

P(S=-1)
P(S=-1 | B=1)

P(B=1 | S=-1)

P(B=1)

	S=-1	S=0	S=1
B=-1	19%	27%	5%
B=1	8%	21%	20%

P(S=-1) $= \sum_b P(S=-1 \; B=b)$

$= P(S=-1 \; B=-1) + P(S=-1 \; B=1)$

$= 19 + 8 = 27\%$

27% of the time, twitter unhappy with Bitcoin, on a day when bitcoin is increasing then only 16% of the tweets are unhappy

P(S=-1 | B=1)

$$P(S=-1 | B=1) = \frac{P(S=-1 \; B=1)}{P(B=1)} = \frac{8}{8+21+20} \simeq 16\%$$

P(B=1 | S=-1)

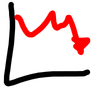
$$= \frac{P(B=1 \; S=-1)}{P(S=-1)} = \frac{8}{19+8} \simeq 29.6\%$$

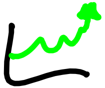
P(B=1)

$= 8+21+20$

$= 49\%$

49% of the time bitcoin grows on a day when twitter is unhappy bitcoin grows only 29.6% of time





	S=-1	S=0	S=1
B=-1	19%	27%	5%
B=1	8%	21%	20%

Conditional Probability (formula version 2: often more useful in our algebraic manipulations)

$$P(a|b) = \frac{P(a\,b)}{P(b)} \longleftrightarrow P(a|b)P(b) = P(a\,b)$$

Takeaway above:

Multiplying

- a conditional probability
- the probability of condition

Will yield

- prob both outcomes happen together

Bayes Rule (Glorified Conditional Probability)

See previous slide

$$P(a|b)\,P(b) = P(ab) = P(b|a)\,P(a)$$

$$\Rightarrow \quad P(a|b) = \frac{P(b|a)\,P(a)}{P(b)}$$

Notice: this formula "swaps" the order of the conditioning: P(A|B) on left P(B|A) on right
Its typical in a Bayes question to be given variables in one order while question asks for other.

A HELPFUL MANIPULATION

MARGINALIZATION

$$P(b) = \sum_a P(a\,b)$$

CONDITIONAL PROB DEFINITION

$$= \sum_a P(b|a)\,P(a)$$

WHY WAS THAT HELPFUL?

BAYES RULE 1

$$P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

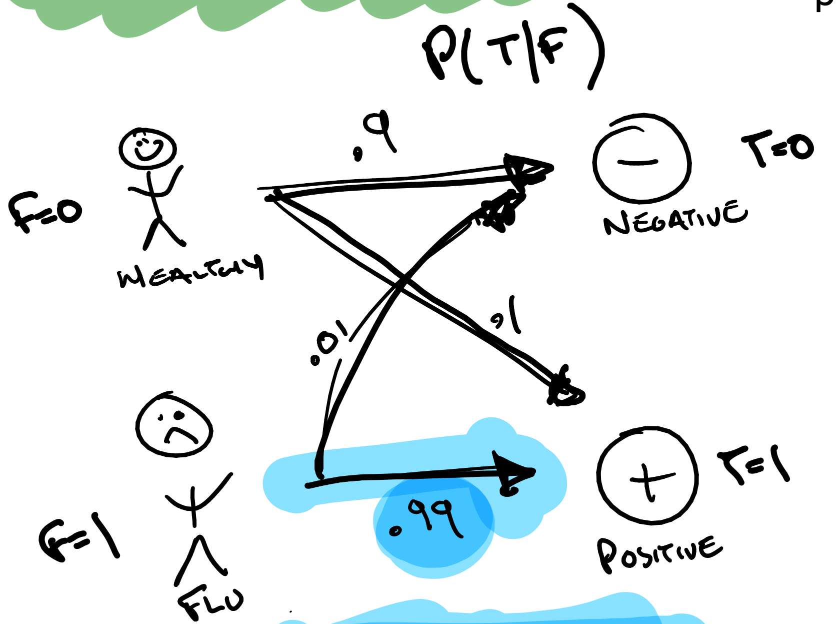
BAYES RULE 2

$$P(a|b) = \frac{P(b|a)\,P(a)}{\sum_i P(b|a_i)P(a_i)}$$

Notice:
all terms of form P(b|a) and P(a) here

Bayes Rule Ex

Given flu occurs in 4% of population, what is the probability one has flu given they test positive?



$$P(T=1|F=1) = .99$$

$$P(F=1|T=1) = \frac{P(T=1|F=1)\,P(F=1)}{P(T=1)}$$

$$\approx \frac{.99 \cdot .04}{.1356}$$

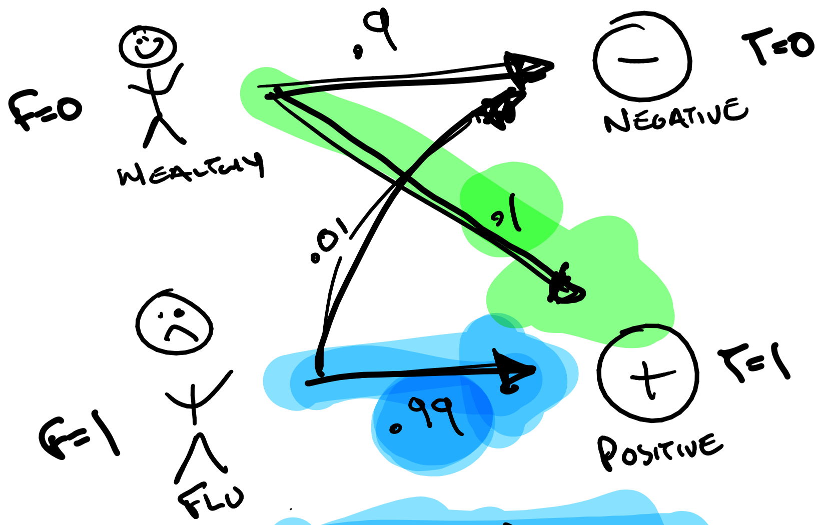
$$\approx .292$$

Bayes Rule Ex

Given flu occurs in 4% of population, what is the probability one has flu given they test positive?

$$P(F=0) = 1 - P(F=1) = 1 - .04 = .96$$

$P(T|F)$



$$P(T=1|F=1) = .99$$

$$P(T=1) = P(T=1\,F=0) + P(T=1\,F=1)$$

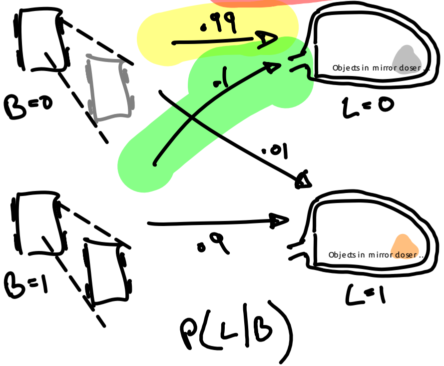
$$= P(T=1|F=0)P(F=0) + P(T=1|F=1)P(F=1)$$

$$= .1 \cdot .96 + .99 \cdot .04$$

$$= .1356$$

In Class Assignment

A blind spot monitor produces a warning light (L=1) when it estimates that a car is in one's blind spot (B=1). Given that the light is off, whats the probability that a car is one's blind spot? (Assume that a car is in your blindspot 2 percent of the time while driving.)



$$P(B=1 \mid L=0) = \frac{P(L=0 \mid B=1)\,P(B=1)}{P(L=0)}$$

$$= \frac{P(L=0 \mid B=1)\,P(B=1)}{P(L=0 \mid B=1)\,P(B=1) + P(L=0 \mid B=0)\,P(B=0)}$$

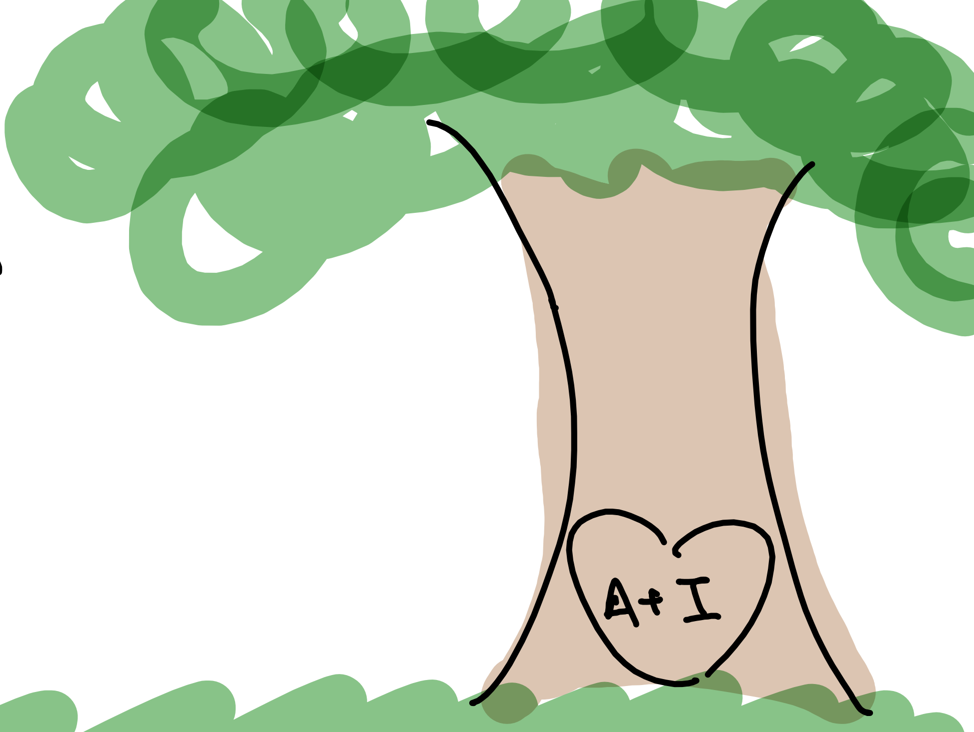
$$= \frac{.1 \cdot .02}{.1 \cdot .02 + .99 \cdot .98} \approx 0.00205719$$

$$P(B=0) = 1 - P(B=1)$$
$$= 1 - .02$$
$$= .98$$

ALGEBRA + INTUITION

LOVE STORY

AHEAD...



INDEPENDENCE + CONDITIONAL PROB

INDEPENDENCE

INTUITION:

Random variables x, y are independent if observing any outcome of one doesn't impact our beliefs about the other.

ALGEBRA:

FOR EACH OUTCOME PAIR x, y

$$P(X=x\ Y=y) = P(X=x)P(Y=y)$$

Bayes Rule shows the equivilence of the algebraic and intuitive definitions above!

INDEPENDENCE + CONDITIONAL PROB

INDEPENDENCE

INTUITION:

Random variables x, y are independent if observing any outcome of one doesn't impact our beliefs about the other.

ALGEBRA:

FOR EACH OUTCOME PAIR x, y

$$P(X=x\ Y=y) = P(X=x)P(Y=y)$$

$$P(x|y) = \frac{P(x\,y)}{P(y)} = \frac{P(x)P(y)}{P(y)} = P(x)$$

Notice that P(X|Y) = P(X). Observing Y has no impact on the prob of X!