Day 10:

Admin:
- plan for Friday: practice exam for 1st half, student-motivated questions in the 2nd half
- exam instructions available (see piazza post)
- hw4 note:
    - final value in counting problem need not be computed explicitly, leave expression (per HW instructions)
- hw4 dates:
    - due Friday @ 11:59 PM
    - late due date is Saturday @ 11:59 PM
    - solutions are available Sunday @ 12:10 AM

Content:
- combinations
- leftover principle
- counting partitions of identical objects

Over-counting (multiplicative)

How many people are in the room if …

… there are 100 eyes in the room
… there are 90 fingers in the room
… there are 400 limbs (legs & arms) in the room

Punchline:
If there are n items (eyes, fingers, limbs)
and c items per every item-of-interest (people)
then there are n / c items of interest

Ordering: when does it matter?

Order matters:

How many ways can a student take 3 CS courses from 10 unique courses?

(CS 1800, DS 2000, DS 2500)

$\neq$

(DS 2500, DS 2000, CS 1800)

← TUPLE

Order doesn't matter:

How many ways can one take 3 candies from 10 unique candies?

{CHOC, LOLLY, GUMMY}

=

{LOLLY, GUMMY, CHOC}

← SET

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? C = {1, 2, 3}
(order doesn't matter)

3 WAYS:

{1, 2} {1, 3} {2, 3}

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? $C = \{1, 2, 3\}$
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING

TWO <u>ORDERED</u> CANDIES:

$(1,2)$ $(1,3)$ $(2,3)$

$(2,1)$ $(3,1)$ $(3,2)$

Combination: (intro example)

How many ways can one choose 2 candies from 3 unique candies? C = {1, 2, 3}
(order doesn't matter)

THERE ARE $P(3,2) = \frac{3!}{1!} = 6$ WAYS OF CHOOSING TWO ORDERED CANDIES:

THERE ARE $2! = 2$ WAYS OF ORDERING 2 CANDIES

$(1, 2)$ $(1, 3)$ $(2, 3)$

$(2, 1)$ $(3, 1)$ $(3, 2)$

OVERCOUNTING (MULTIPLICATION) : WAYS OF CHOOSING 2 FROM 3 (ORDER NOT MATTER) = WAYS OF ORDERING 2 FROM 3 (ORDER MATTERS) / WAYS OF ORDERING 2 (ORDER MATTERS)

Combination: definition & formula

- A combination is a subset of objects (order doesn't matter)
  (how many ways can I choose k items from n possible)

- A permutation is an ordering of objects (order matters)
  (how many ways can I order k items from n possible)

$$\underbrace{C(n,k)} = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!}$$

"n CHOOSE K"

$\binom{n}{k}$ AKA BINOMIAL COEFFICIENT

In Class Activity

$P(8,3)/3! = \binom{8}{3}$ WHAT IF WE ASSUME ORDER DOESN'T MATTER

How many ways can the 8 Mario Kart racers form the final podium of 3 winners.
The order of the podium matters.

(M, P, L)
(M, L, P)
(P, M, L)
(P, L, M)
(L, M, P)
(L, P, M)

THERE ARE 3! ORDERINGS OF {MARIO, PEACH, LUIGI}

ORDER MATTERS
NO REPEATS

$$P(8,3) = \frac{8!}{(8-3)!}$$

In Class Activity

How many ways can the teams (mercedes, ferrari, etc) arrange on the podium of 3 winners in a formula 1 race? (assume that each team has at least 3 cars in the race)

example outcome: (1st place: Mercedes, 2nd place: Mercedes, 3rd place: Ferrari)

11 TEAMS

ORDER MATTERS

REPEATS ALLOWED

(M, F, L, ...)

11^3

$\underset{11}{\underline{M}}$ $\underset{11}{\underline{F}}$ $\underset{11}{\underline{L}}$

In Class Activity

How many 5 card hands exist in a deck of 52 unique cards? ("hands" are unordered)

How many 47 card hands exist in a deck of 52 unique cards?

Compute a final number for the two problems above, how (and why?) are they related?

ORDER DOESN'T MATTER

NO REPEATS

$$\binom{52}{5} = \frac{52!}{47!\,5!}$$

$$\binom{52}{47} = \frac{52!}{5!\,47!}$$

Combinations: Leftover principle

How many ways can I choose all but 10 student to take out for ice cream from this class of size n?

$$\binom{n}{n-10}$$

How many ways can I choose n - 10 students to leave out of my ice cream party?

$$\binom{n}{10}$$

Combinations: Leftover principle

WAYS OF CHOOSING K OBJECTS FROM n

$$\binom{n}{k} = \binom{n}{n-k}$$

WAYS OF CHOOSING n-K OBJECTS TO "LEAVE BEHIND"

For every selection of k items, there is another selection of n-k items which is not chosen.

Counting: Putting it together (almost ... see later slide for complete version of this table)

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$	MYSTERY (FOR NOW)

HOW MANY WAYS CAN 2 PEOPLE SPLIT 8 SLICES OF PIZZA

$(0,8)$ $(3,5)$ $(6,2)$

$(1,7)$ $(4,4)$ $(7,1)$

$(2,6)$ $(5,3)$ $(8,0)$

PERSON 1 GETS 2 SLICES
PERSON 2 GETS 6

CLAIM

$$\binom{8+1}{1} = \frac{9!}{(9-1)!\,1!} = \frac{9!}{8!} = 9$$

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can two people split k slices of pizza?

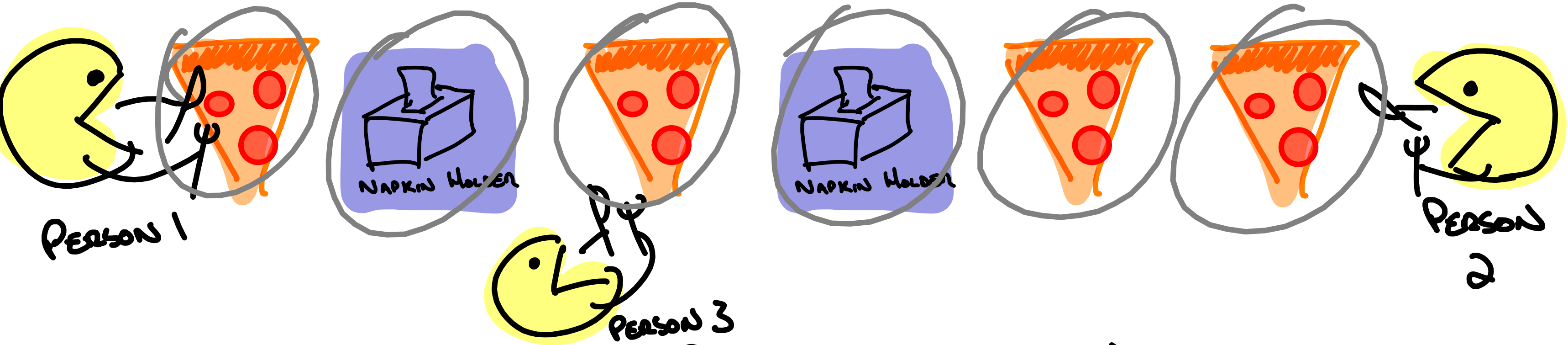
K SLICES PIZZA

1 BOUNDARY

THERE ARE $\binom{K+1}{1}$ WAYS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~two~~ THREE people split K slices of pizza?

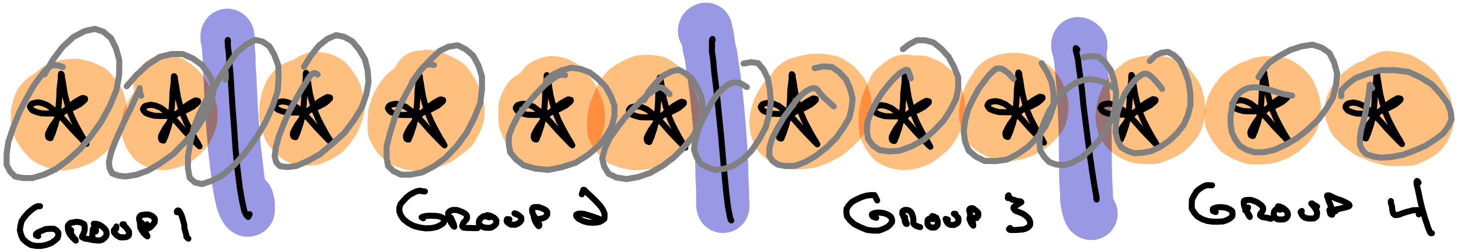


EATS ALL PIZZA BETWEEN BOUNDARY

$$\binom{K+2}{2} \text{ WAYS}$$

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can ~~the people~~ N GROUPS split k ~~s[illegible]~~ STARS?



$$\binom{K+N-1}{N-1} \text{ WAYS}$$

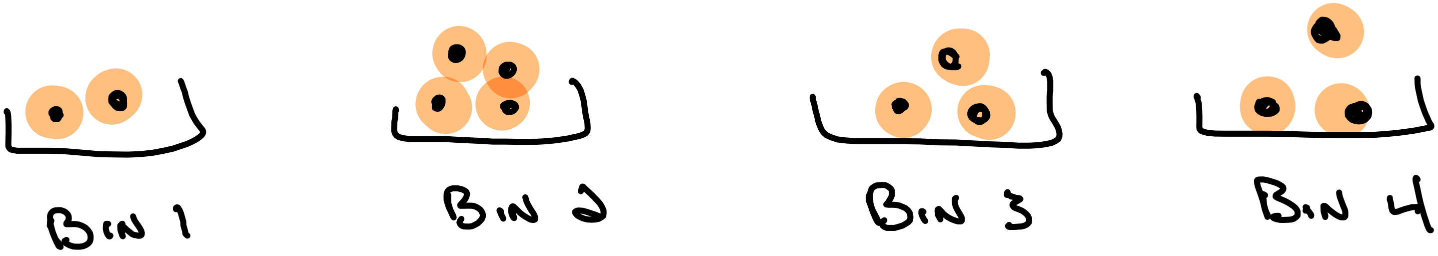
NEED N-1 BOUNDARIES FOR N GROUPS

Counting Partitions of identical objects: (AKA Balls in bins or Stars & bars):

How many different ways can t[redacted] split k s[redacted]?

N BINS

BALLS



$$\binom{K+N-1}{N-1}$$ WAYS

Something is still missing in our chart

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$	PRODUCT RULE N^K
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!K!}$	MYSTERY (FOR NOW)

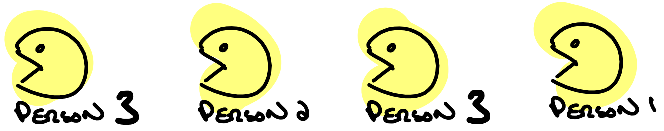
How is the balls-in-bins fit into bottom right box of "putting it together"?

Selecting k items from N items

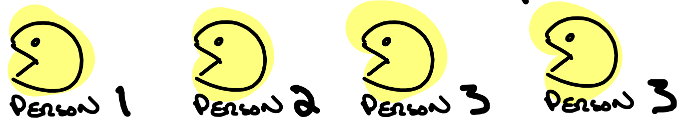
- repeat selections allowed
- order of selections doesn't matter

N ITEMS:

K SELECTIONS: ONE PER SLICE OF PIZZA



OR EQUIVALENTLY



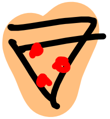
EQUIVALENTLY

HOW TO SELECT K ITEMS FROM N

	NO REPEAT SELECTIONS	REPEAT SELECTIONS
ORDER MATTERS	PERMUTATIONS $P(N,K) = \frac{N!}{(N-K)!}$ How many tuples of length k can one make from N items? (no repeats)	PRODUCT RULE N^K How many tuples of length k can one make from N items? (repeats)
ORDER DOESN'T MATTER	COMBINATIONS $\binom{N}{K} = \frac{N!}{(N-K)!\,K!}$ How many sets with k unique items can one make from N items? (no repeats)	PARTITION OF IDENTICAL ITEMS (STARS + BARS / BALLS IN BINS) $\binom{K+N-1}{N-1}$ How many ways can we split k identical items among N groups?

Two conventions for Stars and Bars

In class now:

K items 

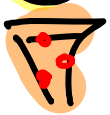
N groups 

$$\binom{K+N-1}{N-1}$$

Appears this way in summary chart

More common:

K groups 

N items 

$$\binom{N+K-1}{K-1}$$

While we're making counting review materials:

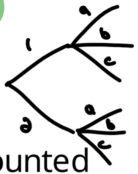
Counting Fundamentals:

- Principle of Inclusion-Exclusion (PIE): Counting the union of sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- Product Rule: How many tuples can be made pulling first item from A and next from B?

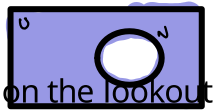
$$|A \times B| = |A| \times |B|$$



Counting moves:

- Count-by-partition: Partition items we want to count into subsets which are more easily counted (remember: each item to be counted shows up in exactly one subset)

- Count-by-complement: Count items not-of-interest, subtract it from "everything"



$$|U - N| = |U| - |N|$$

- Count-by-simplification: Be on the lookout for simpler, equivilent problems

Counting advice:

1. Clearly document your thinking on the paper
   (you'll clarify your thinking and find errors)

2. If you're stuck:
   - head back to the materials of the past few slides (PAGE 18+20)
   - try solving a simpler "sub-problem", the experience may provide fresh insight
      - (often useful for count-by-partition)

In Class Activity:

How many passwords of length 5 can be made from vowels (upper and lowercase)?

5 5

REPEATS ALLOWED: YES

ORDER MATTER: YES

PASSWORD 'ABC'

'CAB'

'AAA'

$N = 10$

$K = 5$

10^5

In Class Activity:

How many ways can I select 10 students in this room to give a million extra credit points to? (assume 200 students in room)

REPEATS ALLOWED: NO

ORDER MATTER: NO

$N = 200$

$K = 10$

$$\binom{200}{10} = \frac{200!}{190!\,10!}$$

In Class Activity:

5 countries each have one woman swimming in the women's 200m freestyle. How many ways might the podium's nationality be arranged? (assume 5 countries each have 1 swimmer each)
(e.g. in tokyo 2020 it was 1. Australia, 2. Hong Kong, 3. Canada)

REPEATS ALLOWED: NO

ORDER MATTER: YES

$N = 5$

$R = 3$

$$P(5,3) = \frac{5!}{(5-3)!} = 5\cdot4\cdot3$$

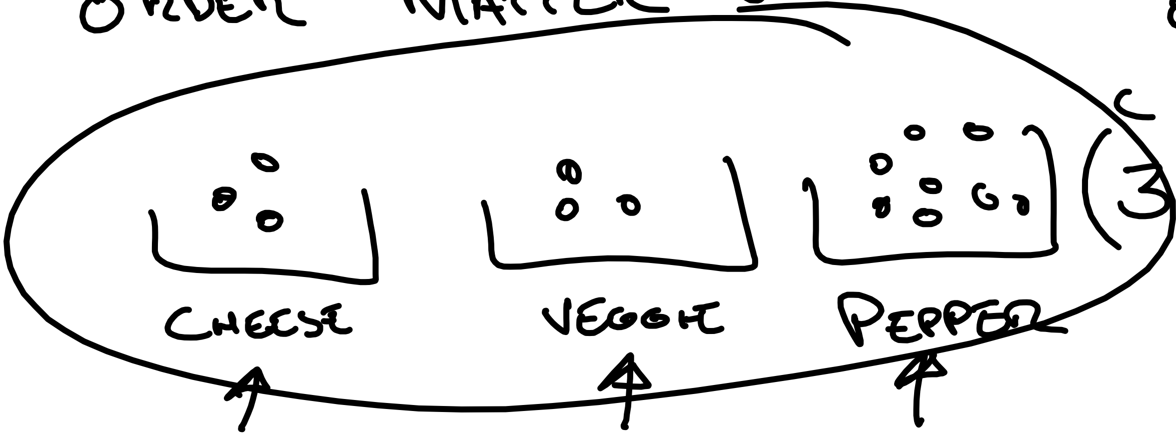
In Class Activity:

How many ways can we order 14 pizza for our TAs from a pizza place which serves 3 types of pizza (cheese, pepperoni, veggie)? Assume a whole pizza may only be of one type.

REPEATS ALLOWED: YES

ORDER MATTER: NO

14 CHEESE
0 PEPPER
0 VEGGIE



C P V
(3, 3, 8)

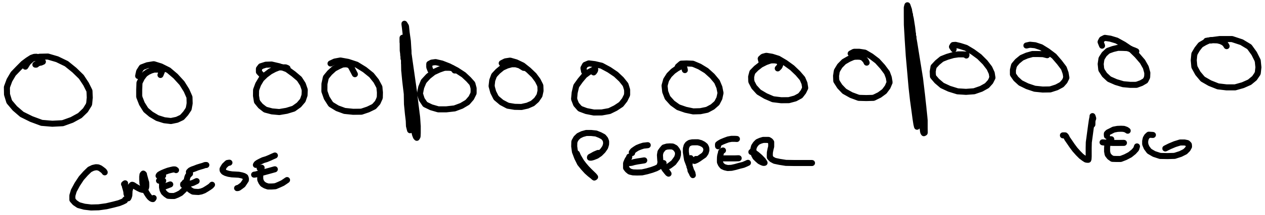
C

CC VVV PPPP
PPPP

$$\binom{14+3-1}{3-1}$$

CCPV

CPVC



I've got 3 pairs of pants, 2 shirts and 5 hats, how many outfits can I wear if I don't wear one pair of pants with one shirt or one hat?

3 P_1 P_2 P_3

2 S_1 S_2

5 H_1 H_2 H_3 H_4 H_5

OUTFITS POSSIBLE − OUTFITS NOT ALLOWED

$$= 3 \cdot 2 \cdot 5 - 5 - 2 + 1$$

$P_1 \cap S_1$ $\quad P_1 \cap H_1$

$\underline{P_1}$ $\underline{\quad}$ $\underline{H_1}$

$$|P \times S \times H| = |P| \times |S| \times |H| =$$

5 COUNTRIES

2 SWIMMERS EACH

PODIUM 3 SPOTS

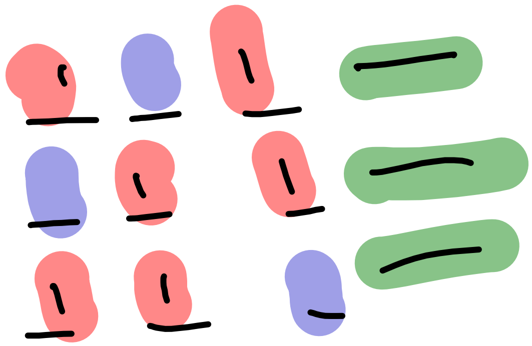
UNIQUE PODIUM + COUNTRY REPEATED

$P(5,3)$

$$\frac{5!}{(5-3)!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = 5\cdot4\cdot3$$

$5\cdot4\cdot3 + 5\cdot4$

$5\cdot4\cdot4$



$5\cdot4\cdot3$

$3 \cdot 5 \cdot 4$