Written Homework 11

Assigned: Thu 12 Apr 2018
Due: Wed 18 Apr 2018

Instructions:

- The assignment has to be uploaded to Blackboard by the due date. NO assignment will be accepted after 11:59pm on that day.

- You may turn in work to Blackboard that is either handwritten and scanned, written in a word processor such as Word, or typeset in LaTeX. In the case of handwritten work, we may deduct points if the scan is upside down or the work is illegible.

- To get full credit, show INTERMEDIATE steps leading to your answers, throughout.

- We expect that you might study with friends and work out solutions to problems together, but you must write up your own solutions, in your own words.

- Some guidelines on collaboration: If you produce a solution together on a whiteboard, don’t simply copy it down afterwards. You must, on your own, write your own solution by hand. If someone explains an answer to you, do not write down their exact words; instead, on your own write up your solution afterwards. In short, your solution should be uniquely yours, and a product of your own understanding.

- If you collaborate with anyone, write their name on the first page of your assignment at the top.

Problem 1 [20 pts (12,8)]: Growth Rates

i. Consider the following fourteen functions for the question that follows:

(a) \( \log_3(2n) \)
(b) \( \sqrt{n} \)
(c) \( n \log_3(n/2) \)
(d) \( \log_2(3n^2) \)
(e) \( 2^n \)
(f) \( 2^n + 2 \)
(g) \( 2^{2n} \)
(h) \( 3n + 5 \log_2(n) \)
(i) \( 5n + \sqrt{n} \)
(j) \( \sum_{k=1}^{n} k = 1 + 2 + 3 + 4 + \ldots + n \)
(k) \( \sum_{k=1}^{2^n} k = 1 + 2 + 3 + 4 + \ldots + 2n \)
(l) \( \sum_{k=1}^{n^2} k = 1 + 2 + 3 + 4 + \ldots + n^2 \)
(m) \( \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + \ldots + n^3 \)
(n) \( \sum_{k=1}^{n} 2^k = 1 + 2 + 4 + 8 + \ldots + 2^n \)

Make a table in which each function is in a column dictated by its \( \Theta \) growth rate. Functions with the same asymptotic growth rate should be in the same column. Columns should be ordered left to right by the rate of growth of their functions: columns with slower growing functions should be to the left of columns with faster growing functions.
Solution: The table below displays the fourteen functions in rows 2, 3, and 4. The header of each column is a simple function that has the same rate of growth as the other functions in the same column. Growth rates increase from left to right.

<table>
<thead>
<tr>
<th>log ( n )</th>
<th>( \sqrt{n} )</th>
<th>( n )</th>
<th>( n \log n )</th>
<th>( n^2 )</th>
<th>( n^4 )</th>
<th>( 2^n )</th>
<th>( 2^{2n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_3(2n) )</td>
<td>( \sqrt{n} )</td>
<td>( 3n + 5 \log_2(n) )</td>
<td>( n \log_3(n/2) )</td>
<td>( \sum_{k=1}^{n} k )</td>
<td>( \sum_{k=1}^{n^2} k )</td>
<td>( 2^n )</td>
<td>( 2^{2n} )</td>
</tr>
<tr>
<td>( \log_2(3n^2) )</td>
<td></td>
<td></td>
<td></td>
<td>( \sum_{k=1}^{2n} k )</td>
<td>( \sum_{k=1}^{n} k^3 )</td>
<td>( 2^{n+2} )</td>
<td></td>
</tr>
</tbody>
</table>

ii. Each blank cell in the table below represents a growth-rate relationship between the function \( f(n) \) labeling its row and the function \( g(n) \) labeling its column. Enter the symbol \( \Theta \) if \( f(n) = \Theta(g(n)) \). Otherwise enter \( O \) if \( f(n) = O(g(n)) \) or enter \( \Omega \) if \( f(n) = \Omega(g(n)) \). Enter nothing if none of these relations applies.

<table>
<thead>
<tr>
<th>( f(n) )</th>
<th>( g(n) )</th>
<th>( n^2 )</th>
<th>( \log n )</th>
<th>( n^2 + n )</th>
<th>( n (\log n)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \log n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n^2 + n )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( n (\log n)^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solution:

<table>
<thead>
<tr>
<th>( f(n) )</th>
<th>( g(n) )</th>
<th>( n^2 )</th>
<th>( \log n )</th>
<th>( n^2 + n )</th>
<th>( n (\log n)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n^2 )</td>
<td>( \Theta )</td>
<td>( \Omega )</td>
<td>( \Theta )</td>
<td>( \Omega )</td>
<td></td>
</tr>
<tr>
<td>( \log n )</td>
<td>( O )</td>
<td>( \Theta )</td>
<td>( O )</td>
<td>( O )</td>
<td></td>
</tr>
<tr>
<td>( n^2 + n )</td>
<td>( \Theta )</td>
<td>( \Omega )</td>
<td>( \Theta )</td>
<td>( \Omega )</td>
<td></td>
</tr>
<tr>
<td>( n (\log n)^2 )</td>
<td>( O )</td>
<td>( \Omega )</td>
<td>( O )</td>
<td>( \Theta )</td>
<td></td>
</tr>
</tbody>
</table>

Problem 2 [16 pts (4,4,8)]: Graphs Trivia

In each question below, either show that the given situation is possible by exhibiting an example, or prove that the situation is impossible.

i. A (undirected) graph has one hundred vertices. Is it possible for its adjacency matrix to have exactly ninety-one 1’s?
**Solution:** No. The diagonal entries are all zero and the non-diagonal entries occur in pairs (due to the symmetry across the diagonal). The number of 1’s is therefore always even and can’t be 91. (Note: the number of 0’s is even if \( n \) is even and odd if \( n \) is odd.)

**ii.** A graph has 10 vertices. Is it possible for it to have

(a) exactly 7 edges?

**Solution:** Yes. Take the graph with vertices 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 in which there is an edge from 8 to each of 1, 2, 3, 4, 5, 6, 7. This graph has exactly seven edges. Note that it is disconnected as 9 and 10 are isolated vertices.

(b) exactly 46 edges?

**Solution:** No. The maximum number of edges is obtained when the graph is complete, that is, when each set of two vertices is an edge. In that case there are exactly \( \binom{10}{2} = 45 \) edges. Thus 46 edges are impossible.

**iii.** Nine students are leaving for summer vacation. If each of them sends a postcard to exactly three of the other eight, is it possible that each student receives cards from the same three students to whom he or she sent cards?

**Solution:** Consider the graph whose vertices are the 9 students and in which there is an edge joining students \( u \) and \( v \) if they exchanged postcards (that is if and only if each sent a postcard to the other).

If each student sends three cards out and receives cards back from each of her recipients, then each student has degree equal to 3. The sum of the degrees is then \( 9 \cdot 3 = 27 \), an odd number. This isn’t possible, thus not every student received cards back from her recipients.

**Problem 3 [24 pts (8,8,8)]: Graph Search Algorithms**

Consider the graph with vertex set \{a, b, c, d, e, f\} and adjacency lists:

- \( a \rightarrow e \rightarrow c \rightarrow f \)
- \( b \rightarrow c \rightarrow f \rightarrow e \rightarrow d \)
- \( c \rightarrow b \rightarrow a \rightarrow e \)
- \( d \rightarrow f \rightarrow b \)
- \( e \rightarrow a \rightarrow b \rightarrow c \)
- \( f \rightarrow a \rightarrow b \rightarrow d \)

**i.** Draw this graph.

**Solution:**

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ii. Starting at vertex \( a \) traverse the graph by \textit{depth-first-search}, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In your drawing, color the tree edges in red and next to each vertex write its place in the order in which the vertices are first visited (with \( a \) as 1).

\textit{Solution:}

iii. Starting at vertex \( a \) traverse the graph by \textit{breadth-first-search}, processing the neighbors of each vertex in the order they appear on the adjacency list of that vertex (thus not in the alphabetical order). In a fresh copy of your drawing of the graph, color the tree edges in blue and next to each vertex write its place in the order in which the vertices are first visited (with \( a \) as 1).

\textit{Solution:}