Written Homework 06

Assigned: Thu 1 Mar 2018
Due: Wed 14 Mar 2018

Instructions:

• The assignment has to be uploaded to Blackboard by the due date. NO assignment will be accepted after 11:59pm on that day.

• We expect that you will study with friends and often work out problem solutions together, but you must write up your own solutions, in your own words. Cheating will not be tolerated. Professors, TAs, and peer tutors will be available to answer questions but will not do your homework for you. One of our course goals is to teach you how to think on your own.

• You may turn in work to Blackboard that is either handwritten and scanned, written in a word processor such as Word, or typeset in LaTeX. In the case of handwritten work, we may deduct points if the scan is upside down or the work is illegible.

• To get full credit, show INTERMEDIATE steps leading to your answers, throughout.

Problem 1 [25 pts (5 each)]: Advanced Counting

Please simplify to exact numbers for our graders.

i. A tournament of 8 champion chess players will begin with 4 one-on-one matches to determine who advances to the next bracket. Each player will play exactly one player in this first round. A match may have a player on white or black, and these are considered different possibilities. However, we consider two matchings to be the same if all four one-on-one matches are the same; these are not ordered. How many different first round setups are there?

Solution: If we imagine the seats being White1, Black1, White2, Black2, ..., White4, Black4, then there are 8! ways to assign players to these seats. But all 4! permutations of the same pairings are the same. So the answer is 8!/4! = 8*7*6*5 = 1680.

ii. I have decided to buy 5 boxes of cookies from a local Girl Scout. The options available are Thin Mints, Quadruple Almond, Peanut Butter Patties, and Caramel deLites. I could buy as many of each kind as I want, but I know I’m buying exactly 5 in total. How many different possible orders of five boxes are there?

Solution: This is a balls-in-bins problem. The 4 cookie types are the bins, and my 5 purchases are the balls. The answer is C(5+3,3) = C(8,3) = 56.
iii. Use Pascal's Triangle to write out the terms of \((x - y)^9\). (You should write Pascal's Triangle out to the appropriate row. Note the minus sign.)

**Solution:** Pascal’s triangle can be found or generated pretty easily, so we won’t reproduce it here. The coefficients are the same as the row that begins 1, 9, 36, . . .: 

\[x^9 - 9x^8y + 36x^7y^2 - 84x^6y^3 + 126x^5y^4 - 126x^4y^5 + 84x^3y^6 - 36x^2y^7 + 9xy^8 - y^9\]

The alternating signs come from the power of \(-y\).

iv. If passwords can consist of upper case letters (26), lower case letters (26), digits (10), or special characters (12), and passwords must contain at least one digit, one letter, and one special character, how many 8-character passwords are there? (You do not need to simplify this answer to a number.)

**Solution:** This requires computing the space of all passwords, then subtracting the number of illegal passwords – and that number itself requires 3-set inclusion-exclusion to compute. The size of the no-letters, no-digits, and no-special-characters password spaces are \(22^8\), \(64^8\), and \(62^8\). The intersections of these sets are all-letters, all-digits, and all-special, of size \(52^8\), \(10^8\), and \(12^8\). There is no intersection of all three sets, because the password has to consist of something; it can’t have no digits, no letters, and no special characters. So the size is \(74^8 - 22^8 - 64^8 - 62^8 + 52^8 + 10^8 + 12^8\).

v. A service offering email addresses has rather strict rules for usernames: the username must contain exactly two dots and have 8 other characters that are lowercase letters; furthermore, the dots can’t be next to each other, nor can they serve as the first or last character of the username. (So, for example, my.cool.id is fine, but this..isbd breaks the adjacent dot rule.) How many usernames are possible with these rules? (You can leave the answer unsimplified.)

**Solution:** The first step is to choose where the dots go, and the easiest way to handle this is as a balls-in-bins problem where three balls (letters) have already been allocated to the bins (spaces between dots). (The balls here are sort of proto-letters that we’ll fill in with actual letters later.) There are \(C(5+2,2)=7*6/2 = 21\) ways to allocate the remaining five letters into the three places (before, between, or after dots). Once the placement of the dots is settled, we just have to fill in the letters (including the letters we pre-allocated). The answer is then \(21 * 26^8\).

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**Problem 2 [25 pts (5 each)]: Basic Probability**

*Settlers of Catan* is a board game about collecting resources such as sheep and wood. Players draw resource cards when the right number on the board is rolled (exactly) as the sum of two six-sided dice. When a 7 is rolled, instead of collecting resources, the player who rolled this play the part of a “robber” and steal a card at random from another player.

(Express probabilities as simplified fractions in your answers.)

i. I will get a sheep card if a total of 11 is rolled on the two dice. What is the probability of that?
Solution: The rolls are (5,6) and (6,5), and there are 36 possible rolls. That’s 2/36 = 1/18.

ii. I will get some resources if the next roll is either 8 (wood) or 11 (sheep). What is the probability of that?

Solution: There are 5 rolls that sum to 8: (2,6),(3,5),(4,4),(5,3),(6,2).
There are 2 that sum to 11, as mentioned previously.
The total probability is then 7/36.

iii. Again, suppose I get a resource if either 8 or 11 is rolled. What is the probability that I get at least one resource in the next two rolls?

Solution: We could get a resource on the first roll (7/36) or the second roll (7/36), but inclusion-exclusion tells us that if we add these probabilities, we must subtract the chance of getting resources on both rolls (7^2/36^2). The result is 7/36 + 7/36 – 7^2/36^2 = (7 * 36 * 2 – 7 * 7)/36^2 = 455/1296.

iv. My opponent has rolled a 7 (robber), and will steal from me. My hand consists of wood, sheep, brick, ore. I shuffle it to prepare for my opponent stealing a card, and after shuffling, I find that strangely, the order is again wood, sheep, brick, ore. What is the probability that shuffling this hand produces exactly the same ordering?

Solution: 4! possible orderings, only one is like that, so 1/4! = 1/24.

v. Suppose I am lucky – my opponent didn’t pick my sheep, and the next roll is an 11, so now I have 2 sheep, a wood, and an ore. But now two other opponents take their turns, they both roll 7’s for the robber, and they both decide to steal from me – and both took my sheep! What was the probability that, in two draws from a hand of {sheep, sheep, wood, ore}, both sheep are randomly taken?

Solution: We can treat the two draws as just a single draw of two cards. If we treat the outcomes as combinations of cards (order doesn’t matter), then only one possible draw of {sheep, sheep} corresponds to this outcome, but there are C(4,2) = 4^2/2 = 6 ways of drawing two cards from four. So there is a 1/6 chance of this outcome.

Another way to look at it is to behave as if order mattered: drawing one card then another, I could draw sheep1 then sheep2 or sheep2 then sheep1, and these are the two possibilities I care about out of 4*3 possible ordered draws. That’s 2/12 = 1/6.

Problem 3 [10 pts]: Pascal’s Triangle

i. Suppose I were to paint Pascal’s triangle on a wall down to some row n (consider the lone 1 at the top to be the 0th row). At row n, I put a bucket where each number is. For each other number in the triangle, I hammer in a nail at that point. The nail is hammered in so precisely that a ball that drops on it has a 50-50 likelihood to hit either the nail below it and to the left, or the nail below it and to the right. It will hit either nail so precisely that there is again a 50-50 chance of going either way on the next level’s nail, until finally, after hitting a nail in each previous row, the ball will drop into one of the buckets on the nth row.
After dropping $2^n$ balls on the top nail, so that each one zigzags down the triangle into a bucket, I find that the number of balls in each bucket roughly matches the Pascal’s Triangle number that I placed the bucket in front of.

Explain why.

*Solution:* Each bucket corresponds to a number of times a particular ball went right instead of left, with the leftmost bucket corresponding to $r = 0$ right bounces and the rightmost bucket corresponding to all $n$ going right. We can think of a ball’s history as the string LRLRRRRRRLLRRR with L encoding “left” and R encoding “right”. The number of histories in which a ball goes right exactly $r$ times is therefore equal to $C(n,r)$, since we are choosing which $r$ characters in the history are R instead of L. That makes the probability that a ball goes into a particular bucket $C(n,r)/2^n$. That means out of $2^n$ balls, we expect about $C(n,r)$ to land in the $r$th bucket. And Pascal’s Triangle gives us exactly the values $C(n,r)$ at each place, so we expect a number of balls in each bucket roughly equal to what it predicts.