Discrete Structures

Lesson 24b - Random Variables

Random Variables

Definition. A random variable $X$ on a sample space $S$ is a function that to each outcome $o$ assigns a real number $X(o)$.

Example 1. Toss a fair coin 3 times. $X =$ number of heads.

possible values of $X$: 0, 1, 2, 3

$X = 0$: T T T

$X = 1$: H T T, T H T, T T H

$X = 2$: T H H, H T H, H H T

$X = 3$: H H H

<table>
<thead>
<tr>
<th>$X$</th>
<th>Pr[$X = 0$]</th>
<th>Pr[$X = 1$]</th>
<th>Pr[$X = 2$]</th>
<th>Pr[$X = 3$]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

Expectation

Definition. The expectation of a random variable $X$ over sample space $S = \{o_1, o_2, ..., o_n\}$ is the weighted sum:

$$E[X] = \frac{1}{n} (X(o_1) + X(o_2) + ... + X(o_n))$$

$E[X]$ is the average value of $X$.

By grouping outcomes with the same $X$-values in

$$E[X] = \frac{1}{n} (X(o_1) + X(o_2) + ... + X(o_n))$$

we obtain

Theorem. The expectation of a random variable $X$ is the sum

$$E[X] = \sum_{a \text{ value of } X} a \cdot \Pr[X = a]$$

Expectation

Problem. Toss a fair coin 3 times. $X =$ number of heads. Find $E[X]$

Answer 1

<table>
<thead>
<tr>
<th>$T$ T $T$</th>
<th>H $T$</th>
<th>T $T$ H</th>
<th>H $H$</th>
<th>T $T$ H</th>
<th>H $H$</th>
<th>$H$ $H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X = 0$</td>
<td>$X = 1$</td>
<td>$X = 1$</td>
<td>$X = 2$</td>
<td>$X = 2$</td>
<td>$X = 2$</td>
<td>$X = 3$</td>
</tr>
</tbody>
</table>

$E[X] = \frac{1}{8} (0 + 1 + 1 + 2 + 2 + 2 + 3) = 3/2$

Answer 2

<table>
<thead>
<tr>
<th>$X = 0$</th>
<th>$X = 1$</th>
<th>$X = 2$</th>
<th>$X = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

$E[X] = 0 \cdot (1/8) + 1 \cdot (3/8) + 2 \cdot (3/8) + 3 \cdot (1/8) = 3/2$
Expectation

Problem. You pay $2 each time to play the following game: Two dice are rolled, and you win $5 for each 6 that comes up. Do you expect to win more than you pay if you play many times?

(a) Define a suitable random variable \(X\).

\(X\) is the gain from the game

(b) Write the distribution of \(X\).

<table>
<thead>
<tr>
<th>(X = 0)</th>
<th>(X = 5)</th>
<th>(X = 10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25/36</td>
<td>10/36</td>
<td>1/36</td>
</tr>
</tbody>
</table>

(c) Compute \(E[X]\).

\[
E[X] = \frac{25}{36} \cdot (25/36) + \frac{5}{36} \cdot (10/36) + \frac{10}{36} \cdot (1/36) = \frac{60}{36} = 2.5
\]

Expected gain is less than $2 invested.

Linearity of Expectation

Example. Toss two fair dice. \(X\) = sum of the numbers. Find \(E[X]\).

(a) Direct approach.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\Pr[X = 2] & \Pr[X = 3] & \Pr[X = 4] & \Pr[X = 5] & \Pr[X = 6] & \Pr[X = 7] & \Pr[X = 8] & \Pr[X = 9] & \Pr[X = 10] & \Pr[X = 12] \\
\hline
\end{array}
\]

\[
E[X] = \frac{1}{36} (2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 + 7 \cdot 6 + 8 \cdot 5 + 9 \cdot 4 + 10 \cdot 3 + 11 \cdot 2 + 12 \cdot 1) = 252/36 = 7
\]

(b) Toss two fair dice. \(Y\) = number on first. \(Z\) = number on second. \(X = Y + Z\).

<table>
<thead>
<tr>
<th>(Y = 1)</th>
<th>(Y = 2)</th>
<th>(Y = 3)</th>
<th>(Y = 4)</th>
<th>(Y = 5)</th>
<th>(Y = 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6 &amp; 1/6 &amp; 1/6 &amp; 1/6 &amp; 1/6 &amp; 1/6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
E[Z] = E[Y] = \sum \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{7}{2}
\]

\[
E[X] = E[Y + Z] = E[Y] + E[Z] = \frac{7}{2} + \frac{7}{2} = 7
\]

Linearity of Expectation

Theorem. If \(X\) and \(Y\) are two random variables on the same probability space, then:

(a) \(E[X + Y] = E[X] + E[Y]\)

(b) \(E[cX] = cE[X]\) for every constant \(c\)

Proof.

(a) \(E[X + Y] = \frac{1}{n} \sum (X + Y(o_i) + \ldots + (X + Y(o_n))\)

\[
= \frac{1}{n} \sum (X(o_1) + \ldots + X(o_n)) + \frac{1}{n} \sum (Y(o_1) + \ldots + Y(o_n))
\]

\[
= E[X] + E[Y]
\]

(b) Similar.